STUDY OF CERTAIN SPECIAL FUNCTIONS FROM THE GROUP-THEORETIC POINT OF VIEW

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Dedicated to

My Mother

&

Sacred Memory

of My

Late Father

DEPARTMENT OF MATHEMATICS
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Certificate

*This is to certify that the dissertation entitled "**STUDY OF CERTAIN SPECIAL FUNCTIONS FROM THE GROUP-THEORETIC POINT OF VIEW**", has been carried out by* **Mr. Sebul Islam Laskar** *under my supervision in the Department of Mathematics, Aligarh Muslim University, Aligarh and the work is suitable for submission for the award of degree of Master of Philosophy in "Mathematics".*

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PREFACE

Special functions and their generating relations arise in a diverse range of applications in harmonic analysis, multivariate statistics, quantum physics, molecular chemistry and number theory.

The majority of functions used in technical and applied mathematics have originated as the result of investigating practical problems. The study of special functions is not only useful for mathematics but for physics, engineering, science, statistics and for technology also.

The beautiful and insufficiently known area of analysis and its connections to group theory is to be used to obtain various properties, recurrence relations, generating functions, integral, representations and symmetry properties of special functions.

Our work in this dissertation is influenced by the work of Erdelyi, Magnus, oberhettinger and Tricomi [42],[43], Weisner [72],[73],[74], Miller [79], [80],[81],[82], Manocha [77], Pathan, Goyal and Shahwan [86][88]. Rainville [92], Srivastava and Manocha [103] and many others.

We study following Miller, that a given class of special functions appears as the set of matrix elements of irreducible representation of a given group. The algebraic properties of the group and its Lie algebra are then reflected in the functional and differential equations satisfied by the given family of special functions. Further, it will be shown that many special functions turn up as the coefficients of the representations of a Lie group and their relations with harmonic analysis on Lie groups are very intimate.

The dissertation comprises of five chapters.

The chapter 1 contains the basis elements of group representation theory with definitions and examples of the Lie groups and Lie algebra and the theory of special functions.

In chapter 2 we have discussed the method of obtaining Bessel differential equation of the first kind of integral order, the related Bessel functions, their generating functions, and some recurrence relations from a Lie-group theoretic approach, the group of transformations being the Euclidean group E_2 for the plane. In addition we have discussed a two-dimensional Helmholtz differential equation which is satisfied by each matrix element of the representation of the translation operator of E_2. Some interesting information can be gained from this differential equation.

In chapter 3 we have discussed some properties of special functions, Rodrigues-type formula and differential equations using some operators defined on a Lie algebra. The analytic methodology developed in the study can easily be adopted to the study of some other special functions of mathematical physics.

In chapter 4, by using a representation of the Lie group T_3 we discussed the method of obtaining generating functions of multivariable generalized Bessel functions (GBF) and then we discussed some of its special cases.

In chapter 5, we explore the method of obtaining certain generating functions by using Lie theoretic method which involves linear differential operators, that forms a 3-dimensional Lie-algebra isomorphic to $sl(2)$. In this chapter we also discussed local multiplier representation $[T(g)f](x,y)$ and after choosing in certain ways this multiplier representation gives the generating functions for Laguerre functions.

CONTENTS

CHAPTER 1

PRELIMINARIES

1.1 INTRODUCTION : Theory of special functions plays an important role in mathematical physics. These functions commonly arise in such areas of application as heat conduction, communication systems, electro-optics, non-linear wave propagation, electromagnetic theory, quantum mechanics, approximation theory, probability theory and electro circuit theory, among others. Special functions are sometimes discussed in certain engineering and physics courses, and mathematical courses like partial differential equations.

The advent of large, fast and sophisticated computing machines did not diminish the importance of special functions within the context of applied sciences. They provide a unique tool for developing simplified yet realistic models of physical problems, thus allowing for analytic solutions and hence a deeper insight into the problem under study. A vast mathematical literature has been devoted to the theory of these functions as constructed in the works of Euler , Gauss, Legendre, Hermite, Riemann , Chebyshev, Hardy, Littlewood, Watson, Ramanujan and other classical authors, for example Erdelyi, Magnus, Oberhettinger and Tricomi [43],[44], McBride[78], Rainville[92], Srivastava and Manocha[103] , Szegö[104], Titchmarsh[107] and Watson [110],[111]. Brief historical synopses can be found in Aksenov[2], Chihara[23], Klein[63], Lavrent'ev and Shabat[68], Miller[82], Whittaker and Watson[114].

One of the numerous consequences of Group theoretic approach to special function is that for obtaining generating functions and other properties of special functions using the theory of contraction and representation of Lie groups. Several papers recently appeared in which Weisner[72] method for obtaining generating function is used. For further example one may refer to the works of Weisner[73],[74], Miller[80], Chatterjea[10], [11], [12], [13], [14], Chen and Feng [16], [46], Chiney [24], [25], Chongdar [26], [27], Chongdar and Chatterjea [28], Das [33], [34], [35], [36], Feng [45], Jain and Agrawal [57], Jain [58], Jain and Manocha [59], Kyriakopoulis[66],[67], Manocha[77], Saha and Chatterjea [15], [95], Vishwanathan[109].

The theory of "Finite and continuous groups" later called Lie groups, was built from about 1873 on by Norwegian mathematician Sophus Lie. It arose out of his work on differential equations and contact transformations, and he had a main goal in mind, namely to develop a Galois theory of differential equations, in which these groups would play the role of the Galois group of an algebraic equation. Lie groups offer, by no means unique, example of a theory created for a certain purpose, but not fulfilling it fully. However, it then went off into many directions.

Lie's approach was analytic, and much influenced by his work on contact transformations. For instance, it seems that an important step for him was the interpretation of the Poisson bracket of two functions as the bracket of two infinitesimal contact transformations. The purely algebraic problems to which his theory led were later solved mostly by other mathematicians, notably by W. Killing and above all by E. Cartan.

The application of group (representation)theory began with the rise of quantum mechanics. H. Weyl and E.P.Wigner through their research work that in 1920's gave the direction and rate of development of representation theory was determined by physics. Physicists aroused interest in the representations of non-compact groups, and their result concerning "physical" transformation groups(E.P.Wigner, 1939,V. Bargmann,1947) suggested ideas of general solution, which were developed in papers by J.M.Gelfand, G.Maikey, Harish Chandra and others.

It is a simple matter now a days to define a real or complex Lie Group as a real or complex analytic manifold G endowed with a group structure such that the map $G \times G \rightarrow G$ given by $(x,y) \rightarrow xy^{-1}$ is analytic. But Sophus Lie couldn't say that and this definition differs from him in two respects. First he considered only transformation group. The notation of Abstract Group was not familiar at that time and even later when it had become more widespread. The second one is that his groups are local infact a neighborhood of the origin in C^n (mostly occasionally R^n). The law of composition was defined for elements sufficiently close to the origin, and given by convergent power series.

This dissertation contains the theory and applications of Lie Groups from the point of view of special functions. The main results and definitions are taken from a wide variety of books, monographs and research papers. The text can be found in the work of Schmid[98], Cohn[31], Lipkin[71], Wolf, Cohen and Dewilde[117], Halgason[50], Humphreys[55], Herman[52], Serre Jeanpierre[100], Hoschschild[53], Belinfante and Kolman [6],Wigner[115], Omori[84], Carter [9], Dewitt and Wheeler [39], Onishchik [85], Samelson[51], Adams[1], Abraham[30], Dornhoff[42], Sagle and Walde[94], Howe[54], Postinikov[90] and Aleksander[3].

For applications of Lie Groups and Lie algebras, special attention has been given to books by authors, Weisner[113], Wigner[116], Tinkham[106], Lyubarskii[75], Wawrzyriczyk[112], Miller[79], Glimore[49], Sattenger[96] and Askey[5].

1.2 LIE GROUPS AND LIE ALGEBRAS:

We are familiar with the concept of a (global) Lie group: A Lie group is both an abstract group and an analytic manifold such that the operations of group multiplication and group inversion are analytic with respect to the manifold structure.

In this section some definitions and examples of Lie groups and Lie Algebra have been mentioned.

A number of excellent books, research notes and lecture notes on

Lie theory and Spacial functions have been published, e.g. Miller[80],Cohn,P.M.[31], Srivastava and Manocha[103], Herman [52], Talman[105]

DEFINITION (Manifold):

Let M be a Haousdorff space. If each point of M has a neighborhood homomorphic to an open set in R^n , then M is called an n-dimensional topological manifold.

DEFINITION (Topological group):

A topological group is a set G with the following properties :

i) G is a group,i.e. there is a multiplication defined on G which satisfies the group axioms

ii) G is a Housdorff space.

iii) The mapping $(x,y) \rightarrow xy^{-1}$ of G x G into G is continuous.

Thus the set G has two structures defined on it, one algebraic and one topological. Algebraically, it is a group and topologically, it is a manifold and they are connected by condition (iii). We express condition (iii) by saying that the topology on G is compatible with the group structure.

DEFINITION(Global Lie Group):

A global Lie group is a set G such that each of the following conditions holds true :

i) G is a group,

ii) G is an analytic manifold

iii) The mapping $(x,y) \rightarrow xy$ of the product manifold G x G into G is analytic.

It follows from the above definition that a global Lie group is essentially a group whose elements can be parametrized analytically.

DEFINITION(Local Lie Group):

Let C^n be the space of complex n-tuples $g=(g_1,g_2,\ldots,g_n)$ where $g_i \in C$ for $i=1,2,\ldots,n$ and define the origin e of C^n by $e=(0,0,0..,0)$

Suppose V is an open set in C^n containing e. A complex n-dimensional local Lie group G in the neighborhood. $V \subset C^n$ is determined by a function $\phi : C^n \times C^n \rightarrow C^n$ such that

1) $\phi(g,h) \in C^n$ for $g,h \in V$.

2) $\phi(g,h)$ is analytic in each of its 2n arguments.

3) If $\phi(g,h) \in V$, $\phi(h,k) \in V$, then $\phi[\phi(g,h),k]=\phi[g,\phi(h,k)]$

4) $\phi(e,g)=g$, $\phi(g,e)=g$ for all $g \in V$.

REMARKS:

1) The dimension of a Lie group is the dimension of the manifold.

2) Every Lie group is a topological group, but every topological group is not a Lie group.

3) A global Lie group in the neighborhood of identity is a local Lie group.

4) The manifold considered for a global Lie group may be replaced by open sets in the case of Local Lie group.

DEFINITION (Analytic homomorphism)

Let G and G' be two Local Lie groups. An analytic homomorphism of a Local Lie group G into a Local Lie group G' is a map $\mu : G \rightarrow G'$, where $\mu(g)$ is defined for g in a suitably small neighborhood W of e, such that

$$\mu\,(gh)=\mu(g)\,\mu(h) \qquad g,h,\ gh \in W, \tag{1.2.1}$$

and μ is an analytic function of the coordinates of G. The group multiplication on the right hand side of (1.2.1) take place in G'.

It follows from (1.2.1) that $\mu(e)=e'$ where é, is the identity element of G' and $\mu(g^{-1})=\mu(g)^{-1}$ for g in a small neighborhood of e in G.

A homomorphism which maps G one to one onto a neighborhood of e' in G' is called an isomorphism. An isomorphism of G onto G is an automorphism. Note that an element h of G defines an automorphism

$$\mu_h(g)=hgh^{-1}, \qquad g \in G \tag{1.2.2}$$

Automorphisms of the form (1.2.2) are called inner automorphisms.

DEFINITION (Tangent Vector):

Let G be a local Lie group and $t \rightarrow g(t) = (\, g_1(t), g_2(t), \ldots, g_n(t)\,)$, $t \in C$, be an analytic mapping of a neighborhood of $0 \in C$ into V such that g(o) = e. We can consider such a mapping to be a complex analytic curve in G passing through e. The tanget vector to g(t) at e is the vector

$$\alpha = (d/dt)g(t)/_{(t=0)} \qquad (1.2.3)$$

$$= (d/dt\, g_1(t), \ldots, d/dt\, g_n(t))/_{t=0} \in C^n$$

Every vector $\alpha \in C^n$ is the tangent vector at e for some analytic curve. In particular, the curve

$$\alpha t = (\alpha_1 t, \alpha_2 t, \ldots, \alpha_n t) \qquad (1.2.4)$$

has the tangent vector

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ at } e.$$

DEFINITION (Commutator):

Let g(t) and h(t) be analytic curves in G with g(0)=h(0)=e and with tangent vectors α and β at e respectively. The curve g(t)h(t) is analytic, and its tangent vector is $\alpha+\beta$ at e (Here the (+) sign refers to the vector addition in C^n).

It follows that the tangent vector of g(t) is α at e, that of $g^{-1}(t)$ is $-\alpha$ at e.

The commutator $[\alpha,\beta]$ of α and β is the tangent vector at e of the analytic curve

$$k(t)=g(\tau)\, h(\tau) g^{-1}(\tau) h^{-1}(\tau), \quad t=\tau^2 \qquad (1.2.5)$$

$$\text{i.e.} \quad [\alpha,\beta]=\frac{d}{d(\tau^2)}[g(\tau)h(\tau)g^{-1}(\tau)h^{-1}(\tau)]/_{\tau=0} \qquad (1.2.6)$$

The commutator has the following properties

i) $[\alpha,\beta]=-[\beta,\alpha]$

ii) $[a\alpha+b\beta,\gamma]=a[\alpha,\gamma]+b[\beta,\gamma]$ where $a,b \in C$

iii) $[[\alpha,\beta]\gamma]+[[\beta,\gamma]\alpha]+[[\gamma,\alpha]\beta]=0$ $\alpha,\beta,\gamma \in C^n$

DEFINITION (Abstract Lie Algebra):

A complex abstract Lie algebra $\mathscr{G}$ of dimension n is a complex vector space of dimension n together with a multiplication $[\alpha,\beta] \in \mathscr{G}$ such that

i) $[\alpha,\beta]=-[\beta,\alpha]$

ii) $[a\alpha+b\beta,\gamma]=a[\alpha,\gamma]+b[\beta,\gamma]$ where $a,b \in C$

iii) $[[\alpha,\beta]\gamma]+[[\beta,\gamma]\alpha]+[[\gamma,\alpha]\beta]=0$ $\alpha,\beta,\gamma \in \mathscr{G}$

DEFINITION (One Parameter Subgroup):

A one parameter subgroup of a Lie group G is a "curve" g(t) in the group with the property that

$$g(t_1+t_2)=g(t_1)g(t_2). \tag{1.2.7}$$

In otherwise, we can say that a one parameter subgroup of a Lie group G is a mapping $t \to g(t)$ of the real numbers in G that is a homomorphism between the additive group of the real numbers and G, i.e. it satisfies (1.2.7) .

The variable t is a real number that serves to label the group elements in the subgroup; it is the "one parameter".

A rather trivial one parameter subgroup is defined by g(t)=e. If

$g(t)$ is a one parameter subgroup, $g(\alpha t)$ is also one parameter subgroup consisting of the same group elements but different in the scale of the parameter.

It is evident from (1.2.7) that $g(0)=e$ and that

$(g(t))^{-1}=g(-t)$.

DEFINITION (Lie Algebra Homomorphism):

Let $\mathcal{G}$ and $\mathcal{G}'$ be two Lie Algebras. A Lie algebra homomorphism from $\mathcal{G}$ to $\mathcal{G}'$ is a map $\tau: \mathcal{G} \to \mathcal{G}'$ such that

i) $\tau(a\alpha+b\beta)=a\tau(\alpha)+b\tau(\beta)$ $\quad a,b \in C$

ii) $\tau([\alpha,\beta])=[\tau(\alpha),\tau(\beta)]$ $\quad \alpha,\beta \in \mathcal{G}$

A Lie algebra homomorphism which is a one to one map of $\mathcal{G}$ onto $\mathcal{G}'$ is called an isomorphism. An isomorphism of $\mathcal{G}$ onto $\mathcal{G}$ is called an automorphism.

DEFINITION (Local Transformation Group):

Let G be an n-dimensional loacl Lie group and U an open set in C^m.

Suppose there is a mapping

$F: U \times G \to C^m$ and write $F(x,g)=xg$ for $x \in U$, $g \in G$

G acts on the manifold U as a local Lie transformation group if the mapping F satisfies the conditions:

1) xg is analytic in the coordinates x and g;

2) $xe=x$,

3) if $xg \in U$ then $(xg)g' = x(gg')$, g, g' and $gg' \in G$

Here e is the identity element of G and $x \in U$ is designated by its coordinates $x=(x_1,\dots\dots\dots,x_m)$.

If $x \in U$ and g is in a sufficiently small neighborhood of e, conditions (2) and (3) yeild the relations $(xg)g^{-1}= x$.

Thus , the map $x \to xg$ is locally one to one for fixed g.

DEFINITION(Lie Derivative):

The Lie derivative $L_\alpha f$ of an analytic function f(x) is

$L_\alpha f(x)=d/dt\,[(\exp \alpha t)f](x)/_{t=0}$, $\alpha \in L(G)$

By direct computation we obtain

$$L_\alpha f(x)=\sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_j p_{ji}(x)\frac{\partial f}{\partial x_i}(x)$$

$$\Rightarrow L_\alpha=\sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_j p_{ji}\;(x)\frac{\partial}{\partial x_i}$$

$$\text{where } p_{ji}(x)=\frac{\partial F_i}{\partial g_j}(x,g)/_{g=e}$$

The commutator $[L_\alpha, L_\beta]$ of the Lie derivatives

L_α , L_β is defined by

$$[L_\alpha,L_\beta]= L_\alpha L_\beta - L_\beta L_\alpha$$

DEFINITION (Local Multiplier representation):

Let G be a local Lie transformation group acting on an open neighborhood U of C^m, $0 \in U$ and let Cl be the set of all complex valued functions on U analytic in a neighborhood of O. A (local) multiplier representation T^{ν} of G on Cl with multiplier ν, consisting of a mapping $T^{\nu}(g)$ of Cl onto Cl defined for $g \in G$, $f \in Cl$ by

$[T^{\nu}(g)f](x)=\nu(x,g)f(xg), x \in U$, where $\nu(x,g)$ is a complex valued function analytic in x and g, such that

i) $\nu(x,e)=1$, for all $x \in U$,

ii) $\nu(x,g_1g_2)=\nu(x,g_1)\,\nu(xg_1,g_2)$, $\quad g_1,g_2,\ g_1g_2 \in G$

Property (ii) is equivalent to the relation

$$[T^{\nu}(g_1g_2)f](x)=[T^{\nu}(g_1)(T^{\nu}(g_2)f](x)$$

DEFINITION (Generalised Lie Derivative):

The generalised Lie derivative $D_{\alpha}f$ of an analytic function f(x) under the 1- parameter group exp αt is the analytic function

$$D_{\alpha}f(x)=d/dt\,[T^{\nu}(\exp \alpha t)f](x)/_{t=0} \tag{1.2.8}$$

For $\nu=1$ the generalised Lie derivative becomes the ordinary Lie derivative.

By direct computation from(1.2.8) we obtain

$$D_{\alpha}f(x)=\sum_{j=1}^{n}\sum_{i=1}^{m}\alpha_j p_{ji}(x)\frac{\partial f}{\partial x_i}(x)+\sum_{j=1}^{n}\alpha_j p_j(x)f(x) \tag{1.2.9}$$

where $p_{ji}(x) = \frac{\partial F_i}{\partial g_j}(x,g)/g=e$

and are analytic and the analytic functions P_j (x) are defined by

$$\sum_{j=1}^{n} \alpha_j p_j(x) = \frac{d}{dt}(\nu(x, \exp \alpha t)) \;/_{t=0}$$

1.3 EXAMPLES OF LIE GROUPS AND LIE ALGEBRAS:

Example(1.3.1):

The n x n complex general linear group GL(n,C) is a set of all n x n nonsingular matrices having their entries in C. The group operation being matrix multiplication. We can write the element $X \in GL(n,C)$ as

$$X = \begin{pmatrix} 1+x_{11} & x_{12} & \cdots\cdots & x_{1n} \\ x_{21} & 1+x_{22} & \cdots\cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots\cdots & 1+x_{nn} \end{pmatrix}$$

$$= E_n + X_{ij}, \quad 1 \leq i, j \leq n \qquad (1.3.1)$$

where E_n is the corresponding unit matrix. Identifying E_n with the origin e of C^{n^2}, we parametrize X in the form

$$X = < X_{ij} > = (X_{11}, \ldots\ldots, X_{ij}, \ldots\ldots X_{nn}) \in C^{n^2} \qquad (1.3.2)$$

The fact that X is non singular necessitates that $< X_{ij} >$ be in a suitably small neighbourhood, say U, of e. Let $X = < X_{ij} >$, $Y = < Y_{ij} >$ and $Z = < Z_{ij} >$. We can find a neighborhood V of e, $V \subset U$, such that, for $< X_{ij} >, < Y_{ij} > \in V$, the parameters Z_{ij} are analytic functions of X_{ij} and Y_{ij}. With respect to the neighbourhood V, the associative property of the local Lie group can easily be verified.

Thus GL(n,C) is an n^2 - dimensional complex local Lie group. GL(n, C) is also a global Lie group.

In particular, the 2 x 2 complex general linear group GL(2,C) (sometimes denoted by G(1, 0)) is set of all 2 x 2 nonsingular matrices).

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a, b, c, d \in C \text{ and } ad - bc \neq 0 \qquad (1.3.3)$$

where the group operation is matrix multiplication. Clearly, the identity element of GL(2, C) is the matrix $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In a neighbourhood of e in GL(2, C), we can introduce coordinates for the element g by setting $g \equiv (g_1, g_2, g_3, g_4) \equiv (a-1, b, c, d-1)$

(1.3.4)

With this coordinate system it is verify that GL(2, C) is a four-dimensional complex local Lie group. These coordinates are valid only for g in a suitably small neighbourhood of e, they cannot be extended overall of C^4. GL(2, C) is also a global Lie group.

Let g(t), g(0) =e be an analytic curve in GL(2) with tangent vector at e given by

$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (d/dt)\, g(t)/_{t=0}$

$\quad = d/dt\, (a(t) - 1, b(t), c(t), d(t))/_{t=0}$

We can identify α with the complex 2 x 2 matrix

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = d/dt \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}_{/\, t=0} \qquad (1.3.5)$$

and L [GL(2)] = $gl(2)$ with the space of all 2 x 2 complex matrices α. In terms of this identification , an explicit calculation gives

$$[\alpha, \beta\,] = \alpha\beta - \beta\alpha \qquad (1.3.6)$$

for α, $\beta \in gl(2)$, where the multiplication on the right hand side of (1.3.6) is matrix multiplication.

The special elements j^+ , j^- , j^3 , ε

$$J^+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} j^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} j^3 = \begin{pmatrix} ½ & 0 \\ 0 & -½ \end{pmatrix}$$

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for $gl(2)$ in the sense that every $\alpha \in gl(2)$ can be written uniquely in the form

$$\alpha = a_1\, j^+ + a_2\, j^- + a_3\, j^3 + a_4\, \varepsilon \ , \qquad a_1, a_2, a_3, a_4 \in C$$

These basis elements obey the commutation relations

$$[j^3, j^-\,] = j^+ \ , \ [j^3, j^-\,] = -j^- \ , \ [j^+, j^-\,] = 2j^3 \ ,$$

$$[\varepsilon\, , j^3\,] = [\varepsilon, j^+\,] = [\varepsilon, j^-\,] = 0 \qquad (1.3.7)$$

where 0 is the 2 x 2 matrix all of whose components are zero.

EXAMPLE (1.3.2):

The special linear group SL(n, C) defined by

$$SL(n, C) = \{ X \in GL(n, C) : \det X=1\} \qquad (1.3.8)$$

is an (n^2 -1) dimensional complex local Lie group.Clearly, SL(n, C) is a subgroup of GL(n, C).

In particular, the 2 x 2 complex special linear group SL(2, C) is the abstract matrix group of all 2 x 2 non singular matrices.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , a, b, c, d \in C \qquad (1.3.9)$$

such that det g = 1. Clearly , SL(2, C) is a subgroup of GL(2,C). We can introduce coordinates for a group element g in a neighbourhood of the identity e of SL(2, C) by setting

$$g \equiv (g_1, g_2, g_3) = (a - 1 , b, c)$$

where , d = (1+ bc)/a . In terms of these coordinates SL(2, C) is a three - dimensional local Lie group. It is also a global Lie group.

Suppose g(t), g(0) = e, is an analytic curve whose tangent vector at e is

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = (d/dt) (a(t) -1, b(t), c(t))/_{t=0}$$

α can be defined with the complex 2 x 2 matrix

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & -\alpha_1 \end{pmatrix} = d/dt \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}_{/\, t = 0} \qquad (1.3.10)$$

since $\frac{d}{dt}\left(\frac{1+b(t)c(t)}{a(t)}\right)_{/t=0} = -\alpha_1$

Thus, L[SL(2)] = $sl(2)$ is the space of all 2 x 2 complex matrices with trace zero. As in our first example the Lie product is given by $[\alpha, \beta] = \alpha\beta - \beta\alpha$ for $\alpha, \beta \in sl(2)$. The elements

$$j+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \qquad j- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad j3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{1.3.11}$$

satisfying the commutation relations

$$[j^3, j^-] = j^+ \ , \ [j^3, j^-] = -j^- \ , \ [j^+, j^-] = 2j^3 \ , \tag{1.3.12}$$

form a basis for $sl(2)$.

Since SL(2) is of great importance in special function theory we will study it in somewhat more detail than GL(2).

For $g \in SL(2)$, $\alpha \in sl(2)$, equations

$$\frac{dg_i}{dt} = \sum_{j=1}^{n} \alpha_j F_{ij}(g), i = 1,2,......,n \tag{1.3.13}$$

can be written in the matrix form

$$\frac{dg}{dt} = \alpha g, \ g(0) = e \tag{1.3.14}$$

The solution of this equation is

$$g(t) = e^{\alpha t} = \sum_{k=0}^{\infty} \frac{\alpha^k t^k}{k!} \tag{1.3.15}$$

which converges for all $t \in C$. Thus, in this case exp αt = $e^{\alpha t}$ and the map $\alpha \rightarrow e^{\alpha}$ is an analytic diffeomorphism of a neighbourhood of $0 \in sl(2)$ onto a neighbourhood of $e \in SL(2)$.

Some particular examples of this mapping are

$$\exp\, aj^3 = \begin{pmatrix} e^{a/2} & 0 \\ 0 & e^{-a/2} \end{pmatrix}, \quad \exp\, bj^+ = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix},$$

$$\exp\, cj^- = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}$$

$$a,\ b,\ c \in C \qquad (1.3.16)$$

Corresponding to any $g \in SL(2)$ define the inner automorphism μ_g of $SL(2)$ by

$$\mu_g\,(h) = gh\, g^{-1}\,,\ h \in SL(2) \qquad (1.3.17)$$

where g^{-1} is the matrix inverse to g.

μ_g induces an automorphism $\mu_g{}^*$ of $sl(2)$ where

$$\mu_g{}^*\,(\alpha) = g\,\alpha\, g^{-1},\ \alpha \in sl(2) \qquad (1.3.18)$$

$\alpha = x_1\, j+ +x_2\, j- +x_3\, j^3$

$$= \begin{pmatrix} x_3/2 & -x_1 \\ -x_2 & -x_3/2 \end{pmatrix}$$

Direct computation shows that the action of $\mu_g{}^*$ on α is

$$\mu_g{}^*\,(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_3/2 & -x_1 \\ -x_2 & -x_3/2 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} \left(\dfrac{ad+bc}{2}\right) - bdx_2 + ac\, x_1 & -ab\, x_3 + b^2\, x_2 - a^2\, x_1 \\ & \\ cd\, x_3 - d^2\, x_2 + c^2\, x_1 & \left(\dfrac{-ad-bc}{2}\right) x_3 + bc\, x_2 - acx_1 \end{pmatrix} \quad (1.3.19)$$

Remarks:

(i) GL(2, C) is a complex general linear group.

(ii) GL(2, R) is a real general group.

(iii) GL(2, C) is of dimension 4 as a manifold.

(iv) GL(2, C) is a Lie group , its dimension is 4.

Example (1.3.3):

The abstract group G(0, 1) consists of all 4 x 4 matrices of the form

$$g = \begin{pmatrix} 1 & c\, e^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; a,b,c,\tau \in C \quad (1.3.20)$$

where the group operation is matrix multiplication . It is easy to verify that G(0, 1) is a group.

The inverse element of g in G(0, 1) is given by

$$g^{-1} = \begin{pmatrix} 1 & -c & -a+bc & -\tau \\ 0 & e^{-\tau} & -e^{-\tau}b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G(0, 1) \tag{1.3.21}$$

and

$$g_1\, g_2 = \begin{pmatrix} 1 & (c_1 + e^{-\tau_1} c_2)\, e^{\tau_1 + \tau_2} & a_1 + a_2 + c_1 b_2\, e^{\tau_1} & \tau_1 + \tau_2 \\ 0 & e^{\tau_1 + \tau_2} & b_1 + e^{\tau_1}\, b_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G(0, 1) \tag{1.3.22}$$

where g_1, g_2 are matrices of the form (1.3.20).

We can introduce coordinates for the element g in G(0,1) by setting

$$g \equiv (a, b, c, \tau) \tag{1.3.23}$$

Thus G(0, 1) is a complex 4- dimensional Lie group. In this case the coordinates (1.3.23) are valid over the entire group and not just in a neighbourhood of the identity . The group G(0, 1) is said to be simply connected.

The Lie algebra L[G (0, 1)] can be identified with the space of 4 x 4 matrices of the form.

$$\alpha = \begin{pmatrix} 0 & x_2 & x_4 & x_3 \\ 0 & x_3 & x_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_1, x_2, x_3, x_4 \in C \qquad (1.3.24)$$

with Lie product $[\alpha, \beta] = \alpha\beta - \beta\alpha$. $\alpha, \beta \in L[G(0, 1)]$.

The matrices

$$j+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad j- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$j3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with commutation relations

$$[j3, j+] = j+ \quad , [j3, j-] = -j- \quad , [j+, j-] = -\varepsilon$$

$$[\varepsilon, j+] = [\varepsilon, j-] = [\varepsilon, j3] = 0 \qquad (1.3.25)$$

where 0 is the 4x4 zero matrix, form a basis for L[G(0,1)].

The exponential map exp α, $\alpha \in L[G(0,1)]$ takes the form-

$$\exp\alpha = e^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}$$

and in this case it is an analytic diffeomorphism mapping of all $L[G(0,1)]$onto $G(0,1)$. In particular,

$$\exp \tau j^3 = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{\tau} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \exp bj+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\exp cj- = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \exp a\varepsilon = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$a,b,c,\tau \in C \qquad (1.3.26)$$

If $g \in G(0,1)$, the inner automorphism μ_g of $G(0,1)$ defined by $\mu_g(h)=ghg^{-1}$, induces an automorphism $\mu_g{}^*$ of $L[G(0,1)]$ where $\mu_g{}^*(\alpha)= g\,\alpha\,g^{-1}$ for $\alpha \in L[G(0,1)]$.

If $\alpha = x_1 j+ + x_2 j- + x_3 j3 + x_4\varepsilon$, direct computation yield

$$\mu_g{}^*(\alpha)=(x_1e^{\tau}-x_3b)j^+ + (x_2e^{-\tau}+x_3c)j^- + x_3j^3+(x_1ce^{\tau}-x_2be^{-\tau} -x_3bc+x_4)\varepsilon \qquad (1.3.27)$$

where g is given by (1.3.20)

Example (1.3.4)

The matrix group T_3 is the set of all 4 x 4 matrices of the form

$$g = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-\tau} & 0 & c \\ 0 & 0 & e^{\tau} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b, c, \tau \in C \tag{1.3.28}$$

The inverse of $g \in T_3$ is given by

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\tau \\ 0 & e^{\tau} & 0 & -e^{\tau}c \\ 0 & 0 & e^{-\tau} & -e^{-\tau}b \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a, b, c, \tau \in C \tag{1.3.29}$$

and the product $g_1\, g_2$ is given by

$$g_1\, g_2 = \begin{pmatrix} 1 & 0 & 0 & \tau_1 + \tau_2 \\ 0 & e^{-\tau_1 - \tau_2} & 0 & c_1 + e^{-\tau_1} c_2 \\ 0 & 0 & e^{\tau_1 + \tau_2} & b_1 + e^{\tau_1} b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.3.30}$$

where g_1 and g_2 are matrices of the form (1.3.28). We can establish a coordinate system for T_3 by assigning to $g \in T_3$ the coordinates

$$g \equiv (\, b, c, \tau) \tag{1.3.31}$$

T_3 is clearly a 3 - dimensional complex local Lie group. Moreover, the coordinates (1.3.31) can be extended over all of C^3. Thus, T_3 has the topology of C^3 and is simply connected.

$\mathcal{T}_3 = L(T_3)$ can be identified with the space of matrices of the form

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & x_3 \\ 0 & -x_3 & 0 & x_2 \\ 0 & 0 & x_3 & x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad x_1, x_2, x_3 \in C \tag{1.3.32}$$

where the Lie product is $[\alpha,\beta] = \alpha\beta - \beta\alpha$, $\alpha,\beta \in \mathcal{T}_3$

A basis for $\mathcal{T}3$ is provided by the matrices

$$j+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad j- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$j3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.3.33}$$

with commutation relations

$$[j3, j+] = j+ \quad , [j3, j-] = -j- \quad , [j+, j-] = 0 \tag{1.3.34}$$

the mapping $\alpha \rightarrow \exp \alpha$ of $\mathcal{T}_3$ onto T_3 is $\exp \alpha = e^{\alpha}$.

Some particular cases of interest are

$$\exp \tau j^3 = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-\tau} & 0 & 0 \\ 0 & 0 & e^{\tau} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \exp bj+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\exp cj- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.3.35}$$

The map μ_g^* defined for $g \in T_3$ by $\mu_g^*(\alpha) = g\alpha g^{-1}$ for all $\alpha \in \mathcal{T}_3$ is an automorphism of $\mathcal{T}_3$. If $\alpha = x_1 j+ + x_2 j- + x_3 j3$ and g is given by (1.3.28), we have

$$\mu_g^*(\alpha) = (x_1 e^{\tau} - bx_3) j+ + (x_2 e^{-\tau} + cx_3) j- + x_3 j3 \tag{1.3.36}$$

Example (1.3.5):

K_5 is the 5-dimensional complex Lie group with elements $g(q,a,b,c,\tau)$, $q, a, b, c, \tau \in C$ and multiplication law

$$g(q, a, b, c, \tau)\, g(q', a', b', c', \tau')$$

$$= g(q + e^{2\tau} q', a + a' + e^{\tau} cb', b + e^{\tau} b' + 2 e^{\tau} cq', c + e^{-\tau} c', \tau + \tau') \tag{1.3.37}$$

In particular the identity element of K_5 is g(0,0,0,0,0) and the inverse of g (q, a, b, c, τ) is

$$g\left(-q e^{-2\tau}, -a + bc - 2c^2 q, -b e^{-\tau} + 2cq e^{-\tau}, -ce^{\tau}, -\tau\right)$$

The associative law can be verified directly. This group has 5 x 5 matrix realization

$$g(q, a, b, c, \tau) = \begin{pmatrix} 1 & c e^{\tau} & b e^{-\tau} & 2a-bc & \tau \\ 0 & e^{\tau} & 2qe^{-\tau} & b-2qc & 0 \\ 0 & 0 & e^{-\tau} & -c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (1.3.38)$$

where now the group operation is matric multiplication.

It is clear from (1.3.37) that the set of all group elements with $q = 0$ forms a subgroup of K_5 isomorphic to G(0, 1).

$L(K_5)$ or k_5 is the 5-dimensional complex Lie algebra with basis j+,j-, j^3, ε, η and commutation relations

$$[j3, j\pm] = \pm j\pm,\ [j3, \eta] = 2\eta,\ [j-, j+] = \varepsilon \qquad (1.3.39)$$

$$[j-, \eta] = 2j+,\ [j+, \eta] = 0,\ [j\pm, \varepsilon] = [j3, \varepsilon] = [\eta, \varepsilon] = 0$$

Clearly, the 4 dimensional subalgebra of k_5 generated j±, j3, ε is isomorphic to $\mathscr{G}(0, 1)$.

A simple computation using (1.3.37) or (1.3.38) shows that Lie algebra of K_5 is isomorphic to k_5.

Infact we can make the identification

$$g(q, a, b, c, \tau) = \exp(q\eta) \exp(a\varepsilon) \exp(bj+) \exp(cj-) \exp(\tau j^3) \tag{1.3.40}$$

where the elements $j\pm$, $j3$, ε, η generate k_5 and satisfy the commutation relations (1.3.39).

Equation(1.3.40) uniquely determines K_5 as a local Lie group. Moreover as a global group K_5 is simply connected.

1.4 REPRESENTATION THEORY:

Let V be a vector space over the field F. (F is either the real numbers, R or the complex numbers, C). Let $\mathscr{G}$ be a Lie algebra over F and denoted by L(V) the space of all linear operators on V.

A representation of $\mathscr{G}$ on V is a homomorphism $\rho : \mathscr{G} \to L(V)$. That is, ρ satisfies the conditions

(i) $\rho(\alpha) \in L(V)$ for all $\alpha \in \mathscr{G}$,

(ii) $\rho([\alpha,\beta]) = [\rho(\alpha), \rho(\beta)]$,

(iii) $\rho(a\alpha.b\beta) = a\,\rho(\alpha) + b\,\rho(\beta)$ $a,b \in F$, $\alpha, \beta \in \mathscr{G}$.

A subspace W of V is said to be invariant under ρ if $\rho(\alpha)\, w \in W$ for all $\alpha \in G$, $w \in W$. A representation ρ of $\mathscr{G}$ on V is reducible if there is proper subspace W of V which is invariant under ρ , and ρ is irreducible if there is no proper subspace W of V which is invariant under ρ.

REPRESENTATION OF THE LIE ALGEBRAS $\mathscr{G}$(a, b):

Here, we consider the Lie algebra $\mathscr{G}(a, b)$ which for any pair of complex numbers (a,b) is a 4-dimensional complex Lie algebra generated by the basis elements j+, j-, j3 , ε satisfying

$$[j+ , j-] = 2\, a^2\, j3 - b\, \varepsilon, \quad [j3 , j+] = j+ \quad ,[j3 , j-] = - j-$$

$$[j+ , \varepsilon] = [j- , \varepsilon] = [j3 , \varepsilon]=0 \qquad (1.4.1)$$

where 0 is the additive identity element. For special choices of the parameters a, b, $\mathscr{G}(a, b)$ essentially coincides with one of the Lie algebras introduced in section (1.3).

In particular we have the following isomorphisms

$$\mathscr{G}(1,0) \cong sl(2) + (\varepsilon)\ , \ \mathscr{G}(0, 1) \cong L\ [G(0, 1)],$$

$$\mathscr{G}(0, 0) \cong L(T_3) + (\varepsilon) \qquad (1.4.2)$$

where (ε) is the 1-dimensional Lie algebra generated by ε.

We also note that ([80], p - 37 lemma 2.1)

$$\mathscr{G}(a, b) \cong \begin{cases} \mathscr{G}(1,0) & \text{if } a \neq 0, \\ \mathscr{G}(0,1) & \text{if } a=0,\ b \neq 0 \\ \mathscr{G}(0,0) & \text{if } a=b=0. \end{cases} \qquad (1.4.3)$$

Now let ρ be a representation of $\mathscr{G}(a,b)$ on the complex vector space V and set

$$J^+ = \rho\ (j^+)\ ,\ \ J^- = \rho\ (j^-)\ ,\ J^3 = \rho\ (j^3)\ ,\ E = \rho\ (\varepsilon) \qquad (1.4.4)$$

Then ρ being Lie algebra representation, the operators j^+ , j^- , j^3 , E obey the same relations as (1.4.1)

Define the spectrum S of J^3 to be the set of eigen values of J^3. The multiplicity of the eigen value $\lambda \in S$ is the dimension of the eigen space V^λ,

$$V^\lambda = \{ v \in V \,/\, J^3V = \lambda V \}$$

We shall analyze the irreducible representation of $\mathscr{G}(a, b)$ and for each such representation we find a basis of V consisting of eigen vectors of J^3, that is, we shall classify all representations ρ of $\mathscr{G}(a, b)$ satisfying the conditions:

(i) ρ is irreducible (1.4.5)

(ii) Each eigen value of J^3 has multiplicity equal to one. There is a countable basis for V consisting of eigen vectors of J^3.

The basic justification for the above requirements is that they quickly lead to connections between $\mathscr{G}(a , b)$ and certain special functions.

Here, our object is to test all the possibilities of ρ. The following remarks will be helpful for this purpose .

(A) Define the operator $C_{a,b}$ on V by

$$C_{a,b} = J^+ J^- + a^2 J^3 J^3 - a^2 J^3 - b J^3 E \tag{1.4.6}$$

It is easy to check that $C_{a,b}$ commutes with every operator $\rho(\alpha)$, $\alpha \in \mathscr{G}(a,b)$. Thus,

$$[C_{a,b}, J^+] = [C_{a,b}, J^-] = [C_{a,b}, J^3] = [C_{a,b}, E] = 0 \qquad (1.4.7)$$

such that $C_{a,b} = \lambda I$ where I is the identity operator and λ is a constant depending upon ρ.

(B) The spectrum S is connected subset of C.

(C) The representation ρ of $\mathcal{G}(a,b)$ is uniquely determined by the constants λ, μ and the spectrum S of J^3.

We need only to consider the Lie algebras $\mathcal{G}(0, 0)$, $\mathcal{G}(0,1)$ and $\mathcal{G}(1, 0)$. Since $\mathcal{G}(a, b)$ is isomorphic to one of the three of (1.4.3). We have

THEOREM (1.4.1) :

Every representation of $\mathcal{G}(0, 0)$ which satisfies (1.4.5) and for which $J^+ J^- \neq 0$ on V is isomorphic to a representation $Q^\mu(w, m_0)$ defined for μ, w, $m_0 \in C$ such that $w \neq 0$ and $0 \leq \mathrm{Re}\, m_0 < 1$. $S = \{ m_0 + n$: n is an integer$\}$. For each representation $Q^\mu(w, m_0)$ there is a basis for V consisting of vectors f_m, $m \in S$, such that

$$J^3 f_m = m f_m \ , \ E f_m = \mu f_m$$

$$J^+ f_m = w f_{m+1}, \ J^- f_m = m f_{m-1} \qquad (1.4.8)$$

$$C_{0,0} f_m = (J^+ J^-) f_m = w^2 f_m$$

1.5 SPECIAL FUNCTIONS:

There are several number of books in which definitions and examples of some important special functions have been mentioned see for example [4], [80], [97] . Throughout our work we shall find it convinient to employ the Pochammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n = \begin{cases} 1, & \text{if } n=0 \\ \lambda(\lambda+1)\text{---------------}(\lambda+n-1), & \text{if } n=1,2,3\ldots\ldots \end{cases} \tag{1.5.1}$$

Since $(1)_n = n!$, $(\lambda)_n$ may be considered as a generalization of the elemetary factorial; hence the symbol $(\lambda)_n$ is also referred to as the factorial function .

In terms of the Gamma functions , we have

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \lambda \neq 0,-1,-2,\ldots\ldots\ldots \tag{1.5.2}$$

Furthermore, the binomial coefficient may now be expressed as

$$\begin{bmatrix}\lambda \\ n\end{bmatrix} = \frac{\lambda(\lambda-1)\ldots\ldots\ldots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \tag{1.5.3}$$

or, equivalently, as

$$\begin{bmatrix}\lambda \\ n\end{bmatrix} = \frac{\Gamma(\lambda+1)}{n!\Gamma(\lambda-n+1)} \tag{1.5.4}$$

If, in the relationship

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} = (-1)^n(-\lambda)_n$$

λ is changed to α-1, then

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(-1)^n}{(1-\alpha)_n},\alpha\neq 0,\pm1,\pm2......... \tag{1.5.5}$$

Since

$$(\lambda)_{-n}=\frac{(-1)^n}{(1-\lambda)_n},\lambda\neq 0,\pm1,\pm2.........,n=1,2,3,.... \tag{1.5.6}$$

and $(\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n$ (1.5.7)

which, in conjunction with (1.5.6) , gives

$$(\lambda)_{n-k}=\frac{(-1)^k(\lambda)_n}{(1-\lambda-n)_k},0\leq k\leq n \tag{1.5.8}$$

For λ=1, we have

$$(n-k)!=\frac{(-1)^k n!}{(-n)_k},0\leq k\leq n \tag{1.5.9}$$

which may alternatively be written in the form :

$$(-n)_k=\begin{cases}\frac{(-1)^k n!}{(n-k)!},0\leq k\leq n\\ 0 \quad , \quad k<n\end{cases} \tag{1.5.10}$$

THE HYPERGEOMETRIC FUNCTIONS:

The term ' hypergeometric' was first used by Wallis in Oxford as early as 1655 in his work ' Arithmetrica Infinitorum' when referring to any series which could be regarded as a generalization of the ordinary geometric series

$$\sum_{n=0}^{\infty} z^n=1+z+z^2+.......... \tag{1.5.11}$$

Because of the many relations connecting the special functions to each other and to the elementary functions , it is natural to enquire whether more general functions can be developed so that the special functions and elementary functions are merely specializations of these general functions.

General functions of this nature have infact been developed and are collectively referred to as functions of the hypergeometric type .

There are several varieties of these functions, but the most common are the standard hypergeometric function .

Some important results concerning the hypergeometric function had been developed earlier by Euler and others , but it was Gauss who made the first systematic study of the series that define this function. Gauss's work was of great historical importance because it initiated for reaching developments in many branches of analysis not only in infinite series , but also in the general theories of linear differential equations and functions of a complex variable. The hypergeometric function has retained its significance in modern mathematics because of its powerful unifying influence, since many of the principal special functions of higher analysis are also related to it.

The main systematic development of what is now regarded as the hypergeometric function of one variable

$$ {}_2F_1[a,b;c;z]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!}, c\neq 0,-1,-2...... \tag{1.5.12}$$

was undertaken by Gauss in 1812.

In (1.5.12), $(a)_n$ denotes the Pochhammer symbol defined by (1.5.1), z is a real or complex variable , a, b and c are parameters which can take arbitrary real or complex values and $c \neq 0, -1, -2 \ldots$. If c is zero or a negative integer, the series (1.5.12) does not exist and hence the function ${}_2F_1\,[a, b; c; z]$ is not defined unless one of the parameters a or b is also a negative integer such that $-c < -a$. If either of the parameters a or b is a negative integer -m then in this case (1.5.12) reduces to the hypergeometric polynomial defined by

$$ {}_2F_1[-m,b;c;z]=\sum_{n=0}^{m} \frac{(-m)_n (b)_n z^n}{(c)_n n!}, -\infty < z < \infty \tag{1.5.13} $$

By d' Alemberts ratio test, it is easily seen that the hypergeometric series in (1.5.13) converges absolutely with in the unit circle , that is , when $|z| < 1$, provided that the denominator parameter c is neither zero nor a negative integer . Notice , however, that if either or both of the numerator parameters a and b in (1.5.12) is zero or a negative integer, the hypergeometric series terminates, and the question of convergence does not enter the discussion.

Further tests show that hypergeometric series in (1.5.13), when $|z| = 1$, (i.e., on the unit circle), is

(i) absolutely convergent, if $\mathrm{Re}\,(c - a - b) > 0$;

(ii) conditionally convergent, if $-1 < \mathrm{Re}\,(c - a - b) \leq 0$, $z \neq 1$;

(iii) divergent, if $\mathrm{Re}\,(c - a - b) \leq -1$.

${}_2F_1\,[a, b; c; z]$ is a solution, regular at $z = 0$, of the homogeneous second order linear differential equation

$$z(1-z)\frac{d^2u}{dz^2}+[c-(a+b+1)z]\ \frac{du}{dz}-abu=0 \tag{1.5.14}$$

where a, b and c are independent of z. (1.5.14) is called the hypergeometric equaton and has atmost three singularities, 0, ∞, and 1 which are all regular

Confluent Hypergeometric Function :

Since the Gauss function ${}_2F_1$ (a,b;c;z) is a solution of the differential equation (1.5.14) replacing z by z/b in (1.5.14) we have

$$z/b\ (1-z/b)\frac{d^2u}{dz^2}+[c-(1+\frac{1+a}{b})z]\ \frac{du}{dz}-au = 0 \tag{1.5.15}$$

Obviously ${}_2F_1$ [a, b ; c ; z/b] is a solution of (1.5.15)

as $b \to \infty$

$$\lim_{b\to\infty} {}_2F_1\,[a, b\ ; c\ ; z/b] = {}_1F_1\,[a;\ c;\ z]$$

is a solution of differential equation

$$z\ \frac{d^2u}{dz^2}+[c-z]\ \frac{du}{dz}-au = 0 \tag{1.5.16}$$

$${}_1F_1\,[a\ ; c\ ; z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}$$

and is called the conflunt hypergeometric function or Kummer's function given by E. E. Kummer in 1836 [65]. It is also denoted by Humbert's symbol ϕ (a;c;z).

The differential equation (1.5.16) has a regular singularity at z= 0 and an irregular singularity at z = ∞.

GENERALIZED HYPERGEOMETRIC FUNCTION:

The hypergeometric function defined in (1.5.12) has two numerator parameters a and b, and one denominator parameter c. It is a natural generalization to move from the definition (1.5.12) to a similar function with any number of numerator and denominator parameters.

We define a generalized hypergeometric function by

$$ {}_pF_q\left[\begin{matrix} a_1,a_2,\ldots\ldots\ldots\ldots,a_p; \\ b_1,b_2,\ldots\ldots\ldots\ldots,b_q; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_n z^n}{\prod_{j=1}^{q} (b_j)_n n!} \tag{1.5.17} $$

where $(a_i)_n$ is Pochhammer symbol given by

$$ (a_i)_n = \frac{\Gamma(a_i+n)}{\Gamma(a_i)} $$

and p and q are positive integers or zero. The numerator parameters $a_1,a_2 \ldots\ldots\ldots a_p$ and the denominator parameters b_1, $b_2 \ldots\ldots\ldots\ldots b_q$ take on complex values , provided that $bj \neq 0, -1, -2\ldots\ldots\ldots$; $j= 1,2\ldots\ldots\ldots q$

An application of the elementary ratio test to the power series on the right in (1.5.17) shows at once that:

(i) If $p \leq q$; the series converges for all finite z.

(ii) If $p = q+1$; the series converges for $|z| < 1$ and diverges for $|z| >1$.

(iii) If $p> q+1$; the series diverges for $z \neq 0$. If the series terminates , there is no question of convergence , and the conclusions (ii) and (iii) do not apply.

(iv) If p = q+1 , the series in (1.5.17) is absolutely convergent on the circle $|z| = 1$ if

$$\operatorname{Re}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0$$

or, we can say that when p = q+1 , the series converges for z=1 provided that $\operatorname{Re}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0$ and for z = -1 provided that

$$\operatorname{Re}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + 1\right) > 0$$

BESSEL FUNCTIONS:

The German astronomer F. W. Bessel (1784 - 1846) first achieved fame by computing the orbit of Halley's comet. In addition to many other accomplishments in connection with his studies of planetory motion , he is credited with deriving the differential equation bearing his name. It is known, however, that Bessel's equation was first investigated in 1703 by J. Bernoulli, who was studying the oscillatory behaviour of the hanging chain. Infact, Bernoulli solved Bessel's equation by an infinite series that now defines the Bessel function of the first kind. Bessel functions were also met with by Euler and others who were concerned with various problems in mechanics. Nonetheless, it was Bessel in 1824, who carried out the first systematic study of the properties of these functions, and thus they are named in his honour.

Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example , they arise in the study of free vibrations of a circular membrane and in finding the temperature distribution in a circular cylinder. They also occur in electromagnetic theory and numerous other areas of physics and engineering . Infact, Bessel functions occur so frequently in practice that they are undoubtedly the most important functions beyond the elementary ones.

The differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)\, y = 0 \tag{1.5.18}$$

where n is a nonnegative constant , is called Bessel's equation , and its solutions are known as Bessel functions and are given by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \tag{1.5.19}$$

For n not an integer, $J_n(x)$ and $J_{-n}(x)$ are linearly independent. However, if n is an integer then $J_{-n}(x) = (-1)^n J_n(x)$ and $J_n(x)$ is the only solution of (1.5.18) which has regular singularity at $z = 0$. Bessel function $J_n(x)$ may also be defined by means of a generating function , for integral n only.

If $t \neq 0$, then for all finite x

$$\exp\left[(x/2)\,(t - 1/t)\right] = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \tag{1.5.20}$$

INTEGRAL REPRESENTATION FOR BESSEL FUNCTIONS :

There are several integral representation of $J_n(x)$ that are especially useful in practice. Foremost among these is one involving the Bessel function of integral order which is given by

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)d\phi \tag{1.5.21}$$

where n is a positive integer. For n is a negative integer, we may take n = -m, where m is positive , that the required result is

$$\int_0^{\pi} \cos(-m\phi - x\sin\phi)d\phi = \pi J_{-m}(x)$$

where m is positive

GENERALISED BESSEL FUNCTION:

Generalizations of Bessel functions in specific forms are suggested in a natural way by many physical problems. In particular on investigating the mutual absorption of two non parallel classical photon fields through the production of electron pairs or the interaction of an intense coherent photon beam with free electrons. One is naturally led to introduce what are now reported as the one index two variables generalized Bessel functions , formally defined by the series

$$J_n(x, y) = \sum_{l=-\infty}^{\infty} J_l(x)J_{n-2l}(y) \tag{1.5.22}$$

It is worth stressing that the interest in series of the form (1.5.22) has been originally manifested by mathematicians during the first years of this century.

In fact, as far as the applications of generalized Bessel functions are concerned; they frequently arise in problems of quantum electrodynamics and optics when the dipole approximation is inadequate and higher order harmonics appear. Significant examples are represented by the already quoted scattering of loser radiation by free or weakly bounded electrons, the emission of electromagnetic radiation and the generation of betatron harmonics by relativistic electrons passing through magnetic undulators, the multiphoton absorption and emission by quantum systems and also problems connected with the synchrotron frequency in a stationary bucket with in a harmonic cavity.

THE $\begin{pmatrix} m \\ n \end{pmatrix}$-ORDER GENERALIZED BESSEL FUNCTIONS:

The generalized Bessel functions, are characterized by two indices, two variables and one parameter. They are defined by a series of products of ordinary Bessel functions of integer order , which generalizes the expression (1.5.22) namely

$$J_n^{(m)}(x, y; t) = \sum_{l=-\infty}^{\infty} t^l J_l(x) J_{n+ml}(y) \qquad (1.5.23)$$

the parameter t is a complex. It is evident that the series (1.5.22) can be recovered from (1.5.23) by setting t= 1 and m= -2. The function (1.5.23) reduces to the ordinary Bessel functions for x or y equal zero, according to

$$\lim_{x \to 0} J_n^{(m)}(x, y; t) = J_n(y) \qquad (1.5.24a)$$

$$\lim_{y\to 0} J_n^{(m)}(x, y; t) = \begin{cases} t^{-n/m} J_{-n/m}(x), & n/m \text{ integer} \\ 0, & \text{otherwise} \end{cases} \qquad (1.5.24b)$$

The generating function $T^{(m)}(x, y; t; \tau)$ defined as usual by the series

$$T^{(m)}(x, y; t;\tau) = \sum_{n=-\infty}^{\infty} \tau^n J_n^{(m)}(x, y; t) \qquad (1.5.25)$$

or

$$T^{(m)}(x, y; t;\tau) = \exp\{(y/2)(\tau - 1/\tau) + (x/2)(1/\tau^m - \tau^m/t)\} \qquad (1.5.26)$$

which is particularly interesting because the argument of the exponential is the sum of two terms : the first is just the same entering the expression of the generating function for ordinary Bessel functions, the second arising from the presence of J_{n+ml}'s in the definition (1.5.23)

In particular the correspondence with the specific values $t=\exp(i\theta)$ and $\tau= \exp(i\phi)$, the expression (1.5.26) specializes into

$$T^{(m)}(x, y; e^{i\theta}, e^{i\phi}) = \exp\{i[y \sin\phi + x \sin(\theta - m\phi)]\} \qquad (1.5.27)$$

thus immediately providing for the generalised Bessel functions the integral representation

$$J_n^{(m)}(x, y; e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \exp\{i[y\sin\phi + x\sin(\theta - m\phi - n\phi)]\} d\phi \qquad (1.5.28)$$

which can be regarded as a sort of Jacobi Anger expansion for the generalized Bessel functions (1.5.23). The presence of the further trigonometric term with argument $m\phi$ confirms the relevance of generalized Bessel functions in problems where the dipole approximation is inadequate and higher - order harmonics should be taken into account.

THE GENERALIZED BESSEL FUNCTIONS OF INTEGER ORDER

Generalized Bessel functions can be defined by the infinite series representation

$$J_n(x, y) = \sum_{l=-\infty}^{\infty} J_{n-2l}(x)\, J_l(y) \qquad (1.5.29)$$

Most of their properties can be directly derived from (1.5.29). It is easy to prove that

$$J_n(x,0) = J_n(x) \qquad (1.5.30a)$$

and

$$J_n(0,y) = \begin{cases} J_{n/2}(y), & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \qquad (1.5.30b)$$

$$J_n(0,0) = \delta_{n,0}$$

and the generating function

$$\sum_{n=-\infty}^{\infty} t^n J_n(x, y) = \exp\left[(x/2)(t - 1/t) + (y/2)(t^2 - 1/t^2)\right] \qquad (1.5.31)$$

Putting t =1 , one has the following closure relation

$$\sum_{n=-\infty}^{\infty} J_n(x, y) = 1 \qquad (1.5.32)$$

Moreover setting $t = \exp[i\phi]$; one gets the generalized Jacobi- Anger expansion . However it is interesting to get from equation (1.5.31) the following infinite sum on odd and even indices , respectively,

$$\sum_{n=-\infty}^{\infty} t^{2n} J_{2n}(x, y) = \cosh\left[(x/2)(t - 1/t)\right] \exp\left[(y/2)(t^2 - 1/t^2)\right]$$

$$\sum_{n=-\infty}^{\infty} t^{2n+1} J_{2n+1}(x, y) = \sinh[(x/2)(t-1/t)] \exp[(y/2)(t^2-1/t^2)] \quad (1.5.33)$$

which are the analogous of similar expression obtained for the ordinary Bessel functions, namely

$$\sum_{n=-\infty}^{\infty} t^{2n} J_{2n}(x) = \cosh[(x/2)((t^2-1)/t)]$$

$$\sum_{n=-\infty}^{\infty} t^{2n+1} J_{2n+1}(x) = \sinh[(x/2)((t^2-1)/t)] \quad (1.5.34)$$

An integral representation of the Generalized Bessel functions is given by

$$J_n(x,y) = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi - x\sin\phi - y\sin 2\phi)\, d\phi \quad (1.5.35a)$$

In addition to equation (1.5.35a) , one can also obtain the following integral representation

$$J_n(x, y) = \frac{1}{2\pi i} \int^{(0+)} t^{-n-1} \exp[(x/2)(t-1/t) + (y/2)(t^2-1/t^2)]\, dt$$

$$(1.5.35b)$$

where the symbol $\int^{(0+)}$ denotes integration round a contour which encircles the origin once counter clockwise.

I-TYPE MODIFIED GENERALIZED BESSEL FUNCTIONS :

The I- type modified generalized Bessel function is given by

$$J_n(x, y; s) = \sum_{l=-\infty}^{\infty} s^l J_{n-2l}(x) J_l(y) \quad (1.5.36)$$

The generating function is given by

$$T(x, y ;s; t) = \sum_{n=-\infty}^{\infty} t^n J_n(x,y;s) = \exp[(x/2) (t- 1/t) + (y/2) (st^2-1/ st^2)]$$

(1.5.37)

The modified Generalized Bessel function of I - type, namely

$$I_n (x , y ; u) = \sum_{l=-\infty}^{\infty} u^l I_{n-2l}(x) I_l (y) \tag{1.5.38}$$

are linked to J_n (x, y; u) by

$$I_n (ix , iy ; iu) = i^n J_n(x, y; u) \tag{1.5.39}$$

PROPERTIES OF I -TYPE MODIFIED GENERALIZED BESSEL FUNCTION:

$$I_n (x , -y) = I_n (x , y, -1)$$

$$I_n (-x, y) = (-1)^n I_n (x , y) \tag{1.5.40}$$

$$I_{-n} (x, y) = I_n (x , y)$$

$$I_n (-x, -y; -u) = (-1)^n I_n (x , y ; u)$$

where $I_n (x, y; 1) = I_n (x , y)$

Furthermore , the odd and even index sum rules read as

$$\sum_{n=-\infty}^{\infty} t^{2n} I_{2n}(x,y) = \cosh [(x/2) (t+ 1/t)] \exp[(y/2) (t^2+1/ t^2)]$$

$$\sum_{n=-\infty}^{\infty} t^{2n+1} I_{2n+1}(x,y) = \sinh [(x/2) (t+ 1/t)] \exp[(y/2) (t^2+1/ t^2)]$$

(1.5.41)

A two variable one-index Generalized Bessel function is defined by the series

$$J_n(x,y) = \sum_{l=-\infty}^{\infty} J_{n-2l}(x)\, J_l(y) \tag{1.5.42}$$

and the relevant Jacobi - Anger expansion by

$$\sum_{n=-\infty}^{\infty} e^{in\theta} J_n(x,y) = \exp\left[\, i(x \sin\theta + y \sin 2\theta)\right] \tag{1.5.43}$$

A three - variable GBF defined by the series

$$J_n(x_1, x_2, x_3) = \sum_{l=-\infty}^{\infty} J_{n-3l}(x_1,x_2)\, J_l(x_3) \tag{1.5.44}$$

and the Jacobi - Anger expansion by

$$\sum_{n=-\infty}^{\infty} e^{in\theta} J_n(x_1,x_2,x_3) = \exp\left[\, i \sum_{s=1}^{3} x_s \sin s\theta \,\right] \tag{1.5.45}$$

The m- variable extension of the above relations is straight forward and read as

$$J_n(x_1, x_2, \ldots\ldots, x_m) = \sum_{l=-\infty}^{\infty} J_{n-ml}(x_1,x_2,\ldots\ldots,x_{m-1}) J_l(x_m) \tag{1.5.46}$$

and it is worth stressing that in this case we have

$$\sum_{n=-\infty}^{\infty} e^{in\theta} J_n(x_1,x_2,\ldots\ldots,x_{m-1},x_m) = \exp\left[\, i \sum_{s=1}^{m} x_s \sin(s\theta) \,\right] \tag{1.5.47}$$

It follows from equation (1.5.47) that for $\theta = 0$ one has

$$\sum_{n=-\infty}^{\infty} J_n(x_1, x_2, \ldots\ldots\ldots, x_m) = 1 \tag{1.5.48}$$

The sum of the exponent of equation (1.5.47) runs on both even and odd indices.

ONE PARAMETER GENERALIZED BESSEL FUNCTION:

The one parameter generalized Bessel function

$J_n(x, y; t)$ is defined as follows

$$J_n(x, y; t) = \sum_{l=-\infty}^{\infty} t^l J_{n-2l}(x) J_l(y) \quad (x, y) \in R \tag{1.5.49}$$

and for t =1 reduces to the G. B. F. , $J_n(x, y)$ expressed as follows

$$J_n(x, y; 1) \equiv J_n(x, y) = \sum_{l=-\infty}^{\infty} J_{n-2l}(x) J_l(y) \tag{1.5.50}$$

Taking into account the known result $J_m(0) = \delta_{m,0}$, the above $J_n(x, y; t)$ reduces to the usual Bessel function, $J_m(\xi)$ in the following particular cases

$$J_n(0, y; t) = \begin{cases} t^{n/2} J_{n/2}(y) & , \text{ for n even} \\ 0 & , \text{ for n odd} \end{cases} \tag{1.5.51}$$

$$J_n(x, 0; t) = J_n(x) \tag{1.5.52}$$

Moreover one has

$$J_n(x, y; 0) = 0, \tag{1.5.53a}$$

$$J_n(0, 0; t) = \delta_{n,0} \tag{1.5.53b}$$

Here, $\delta_{n,m}$ is the Kronecker - symbol. By analogy with the one parameter Generalized Bessel function $J_n(x, y; t)$, one can define the one parameter modified generalized Bessel function $I_n(x, y; t)$ as follows

$$I_n(x, y; t) = \sum_{l=-\infty}^{\infty} t^l I_{n-2l}(x) I_l(y) \tag{1.5.54}$$

which for t = 1, reduces to the modified generalized Bessel function I_n (x, y) and given by

$$I_n(x, y; 1) = I_n(x, y) = \sum_{l=-\infty}^{\infty} I_{n-2l}(x) I_l(y) \qquad (1.5.55)$$

By means of known result $I_n(0) = \delta_{n,0}$, the above function reduces to the usual modified Bessel function $I_n(\xi)$, in the following cases

$$I_n(0, y; t) = \begin{cases} t^{n/2} I_{n/2}(y), & \text{for n even} \\ 0, & \text{for n odd} \end{cases} \qquad (1.5.56)$$

$$I_n(x, 0; t) = I_n(x) \qquad (1.5.57)$$

In addition one has

$$I_n(x, y; 0) = 0, \qquad (1.5.58)$$

$$I_n(0, 0; t) = \delta_{n,0} \qquad (1.5.59)$$

The function $I_n(x, y; t)$ is related to the function $J_n(x, y; t)$ via the equation

$$I_n(ix, iy; iu) = i^n J_n(x, y; u) \qquad (1.5.60)$$

GENERATING FUNCTION:

Function $J_n(x, y; t)$ has the following generating function

$$T(x, y; \tau; t) = \sum_{n=-\infty}^{\infty} t^n J_n(x, y; \tau) \qquad (1.5.61)$$

$$= \exp\left[(x/2)(t - 1/t) + (y/2)(\tau t^2 - 1/\tau t^2)\right], \quad (\tau, t) \neq 0 \qquad (1.5.62)$$

From equation (1.5.61) and (1.5.62), one can obtain some useful relations for physical applications. More precisely, putting t = 1 in equations (1.5.61) and (1.5.62), we have

$$\sum_{n=-\infty}^{\infty} J_n(x,y;\tau) = \exp\left[(y/2)\left(\tau - 1/\tau\right)\right] \tag{1.5.63}$$

DEFINITION (HERMITE POLYNOMIALS):

Hermite polynomials $H_n(x)$ are defined by means of the generating relation

$$e^{2xt - t^2} = \sum_{n=0}^{\infty} H_n(x)\, t^n/n! \quad \text{valid for all finite x and t.} \tag{1.5.64}$$

The differential equation of the form

$$H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0 \tag{1.5.65}$$

is known as Hermite differential equation .

RODRIGUES FORMULA:

Rodrigues formula for the Hermite polynomial $H_n(x)$ is given by

$$H_n(x) = (-1)^n\, e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \tag{1.5.66}$$

ORTHOGONALITY:

The orthogonality of the Hermite polynomial $H_n(x)$ is given by

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0 & \text{when } m \neq n \\ \sqrt{\pi}\, 2^n\, n! & \text{when } m = n \end{cases} \tag{1.5.67}$$

DEFINITION (LAGUERRE POLYNOMIALS):

Laguerre polynomials $L_n(x)$ of order n are defined by means of a generating relation

$$\frac{\exp\left\{\frac{-xt}{1-t}\right\}}{1-t}=\sum_{n=0}^{\infty} L_n(x)t^n \tag{1.5.68}$$

$L_n(x)$ can also be written in series form as

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r n! x^r}{(r!)^2(n-r)!} \tag{1.5.69}$$

RODRIGUES' FORMULA :

Rodrogues' formula for Laguerre polynomial is given by

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \tag{1.5.70}$$

ASSOCIATED LAGUERRE POLYNOMIALS:

We define , for n a non - negative integer,

$$L_n^k(x)=\sum_{r=0}^{n} \frac{(-1)^r(n+k)!x^r}{(n-r)!(k+r)!r!} \tag{1.5.71}$$

where $L_n^k(x)$ are associated Laguerre polynomials . This is also called generalized Laguerre or Sonine polynomials .

Note: When k = 0, equation (1.5.71) becomes simple Laguerre polynomials given by equation (1.5.69)

DEFINITION (JACOBI POLYNOMIALS):

The Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ of order n is defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left(\begin{matrix}-n,\ 1+\alpha+\beta+n\\ 1+\alpha\end{matrix};\ \frac{1-x}{2}\right) \tag{1.5.72}$$

for Re(α) > -1 , Re(β) > -1 and n being a non negative integer.

The Jacobi polynomials can also be expressed in the hypergeometric form in the following manner

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\left(\frac{x+1}{2}\right)^n\ {}_2F_1\left(\begin{matrix}-n,\ -\beta-n\\ 1+\alpha\end{matrix};\ \frac{x-1}{x+1}\right) \tag{1.5.73}$$

and $P_n^{(\alpha,\beta)}(x)$ in the series form is given by

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(1+\alpha+\beta)_{n+k}\left(\frac{x-1}{2}\right)^k}{k!(n-k)!(1+\alpha)_k(1+\alpha+\beta)_n} \tag{1.5.74}$$

1.6 GENERATING FUNCTIONS

The name 'generating function' was introduced by Laplace in 1812. Since then the theory of generating functions has been developed into various directions and found wide applications in various branches of science and technology. A generating function may be used to define a set of functions, to determine a differential reccurence relation or a pure recurrence relation, to evaluate certain integrals, etc.

LINEAR GENERATING FUNCTION :

Consider a two variable function F(x,t) which posseses a formal (not necessarily convergent fot t $\neq$ 0) power series expansion in t such that

$$F(x,t)= \sum_{n=0}^{\infty} f_n(x)t^n \tag{1.6.1}$$

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of t. Then the expansion (1.6.1) of F(x,t) is said to have generated the set $\{f_n(x)\}$ and F(x, t) is called a linear generating function (or simply, a generating function) for the set $\{f_n(x)\}$.

This definition may be extended slightly to include a generating function of the type :

$$G(x,t) = \sum_{n=0}^{\infty} C_n g_n(x) t^n \qquad (1.6.2)$$

where the sequence $\{C_n\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$, but is independent of x and t.

A set of functions may have more than one generating function.However, if $G(x,t)=\sum_{n=0}^{\infty} h_n(x)t^n$ then G(x,t) is the unique generator for the set $\{h_n(x)\}$ as the coefficient set.

We now extend our definition of a generating function to include functions which possesses Laurent series expansions. Thus, if the set $\{ f_n(x) \}$ is defined for $n=0,\pm1,\pm2,\ldots\ldots\ldots$, the defination (1.6.2) may be extended in terms of the Laurent series expansion :

$$F^* (x,t)= \sum_{n=-\infty}^{\infty} \gamma_n f_n(x) t^n \qquad (1.6.3)$$

where the sequence $\{\gamma_n\}_{n=-\infty}^{\infty}$ is independent of x and t.

BILINEAR AND BILATERAL GENERATING FUNCTIONS:

If a three variable function F(x,y,t) possesses a formal power series expansion in t such that

$$F(x,y,t)= \sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n \qquad (1.6.4)$$

where the sequence $\{\gamma_n\}$ is independent of x,y and t then F(x,y,t) is called a bilinear generating function for the set $\{f_n(x)\}$.

Now, suppose that a three variable function H(x,y,t) has a formal power series expansion in t such that

$$H(x,y,t)= \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n \qquad (1.6.5)$$

where the sequence $\{h_n\}$ is independent of x,y and t, and the sets of functions : $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are different, then H(x,y,t) is called a bilateral function for the set $\{f_n(x)\}$ or $\{g_n(x)\}$.

The above definition of a bilateral generating function, used earlier by Rainville [92] and Mc Bride [78] may be extended to include bilateral generating functions of the type:

$$H(x,y,t)= \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) g_{\beta(n)}(y) t^n , \qquad (1.6.6)$$

where the sequence $\{\gamma_n\}$ is independent of x, y and t, the sets of function $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are different, and $\alpha(n)$ and $\beta(n)$ are functions of n which are not necessarily equal.

CHAPTER 2

LIE GROUP THEORY OF THE BESSEL EQUATION OF THE FIRST KIND OF INTEGRAL ORDER

2.1 Introduction:

Special functions and recurrence relations of mathematical physics (Courant and Hilbert 1953 [32] , Morse and Feshback 1953 [83] , Rainville, 1960 [92], Lebedev, 1965 [70]) have properties which, for the most part, are derived on the basis of the methods of classical analysis. An alternative to this mode of study of functions of mathematical physics is a group - theoretic approach, (Vilenkin, 1968 [108]). This approach elucidates the geometric background of the special functions , such as rotations, translations and others. The group - theoretic approach to the derivation of the properties of the special functions simplifies considerably the complicated mathematical manipulations of power series and integral representations which characterize the study of the classical theory of the special functions. The following is the correspondence between the classical and group theoretic approaches to the study of the special functions of mathematical physics :

(i) The addition theorem of the special functions becomes multiplication laws for the elements of the group of symmetry involved.

(ii) The differential equations satisfied by special functions are obtained as limiting cases of the addition theorems, or as expressions of the fact that multiplication of group elements in the neighbourhood of the identity element furnishes group elements whose properties are in close proximity to the parameters of the elements multiplied.

(iii) The integral relationships among classical special functions now derived from Frobenius' orthogonality relations for the matrix elements of irreducible representations as generalized for Lie groups by means of Hurwitz's invariance integers.

(iv) Lie groups can be considered as limiting cases of others, and this furnishes further relations between them.

For example, the Euclidean group of the plane can be obtained as a limit of the group of rotations in three - space, and so the elements of the representations of the former [Euclidean group of the plane] are limits of the representations of the latter group. While the former group relates to the Bessel equation of the first kind, of integral order, and the associated Bessel functions, the latter group is related to the Jocobi functions. Clearly, elements of certain group representations are specified special functions of mathematical physics.

In this chapter, we discuss the method of obtaining the Bessel equation of the first kind of integral order n, and the associated Bessel (special) functions from the elements of the group representations of the Euclidean group E_2 for the plane. This approach, the Lie - group theoretic approach, provides a good alternative to the conventional series method due to Frobenius. The technique can be extended to the study of other differential equations of mathematical physics and their associated special functions once applicable symmetry groups are found as well as their desired representations. A number of properties of such differential equations and the associated special functions can be obtained group theoretically.

In section (2.2), the properties of the Euclidean group E_2 of the plane and Frobenius method of induced representations are discussed.

In section (2.3), we discuss the method of obtaining the Bessel function J_m of the first kind and of integral order by applying the complete representation of E_2. Also we discuss the method of obtaining some well known recurrence relations of Bessel functions J_m by the use of general addition theorem.

In section (2.4), we discuss the method of obtaining the Bessel differential equation of the first kind of integral order m, its generating function and a Helmholtz partial differential equation satisfied by each matrix element of the representation of the translation operator of E_2.

2.2. PROPERTIES OF THE EUCLIDEAN GROUP E_2 OF THE PLANE AND THE FROBENIUS METHOD OF INDUCED

REPRESENTATION OF E_2:

The Euclidean group E_2 of the plane is the set of all transformations of the plane, of the form

$$T(\bar{a})\ R(\theta) \tag{2.2.1}$$

where $R(\theta)$ is a rotation of the plane about the origin by an angle θ, and $T(\bar{a})$ is a translation of the plane by the vector $\bar{a}$. The coordinates (x', y') of an arbitrary point (x, y) following the transformtion (2.2.1) are given by

$$(x', y') = T(\bar{a})\ R(\theta) \quad \{(x, y)\} \tag{2.2.2}$$

i.e.

$$\begin{aligned} x' &= x \cos\theta - y \sin\theta + a \\ y' &= x \sin\theta + y \cos\theta + b \end{aligned} \tag{2.2.3}$$

or, $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.2.4}$$

where a and b are the components of $\bar{a}$. The three parameters of E_2 are thus a, b and θ and the infinitesimal generators of the Lie

group of the continuous group E_2 or of the corresponding Lie algebra are

$$L_a,\ L_b,\ L_\theta$$

These are calculated in the differential form as follows:

Replace θ, a, b by their infinitesimals $\delta\theta$, δa & δb, respectively, to obtain infinitesimal rotations and translations, namely

$$x' = x \cos \delta\theta - y \sin \delta\theta + \delta a \qquad (2.2.5)$$
$$y' = x \sin \delta\theta + y \cos \delta\theta + \delta b$$

i.e.

$$\text{or,} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x & -y\delta\theta \\ x\delta\theta & y \end{pmatrix} + \begin{pmatrix} \delta a \\ \delta b \end{pmatrix} \qquad (2.2.6)$$

as $\delta\theta \to 0$, Equation (2.2.6) gives,

$$x' - x = \delta x = - y\, \delta\theta + \delta a$$

$$y' - y = \delta y = x\, \delta\theta + \delta b$$

so that

$$\frac{\partial x}{\partial \theta} = -y, \frac{\partial x}{\partial a} = 1, \frac{\partial x}{\partial b} = 0$$

$$\frac{\partial y}{\partial \theta} = x, \frac{\partial y}{\partial b} = 1, \frac{\partial y}{\partial a} = 0$$

and

$$L_\theta = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$L_a = \frac{\partial x}{\partial a}\frac{\partial}{\partial x} + \frac{\partial y}{\partial a}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} \qquad (2.2.7)$$

$$L_b = \frac{\partial x}{\partial b}\frac{\partial}{\partial x} + \frac{\partial y}{\partial b}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}$$

which are the three infinitesimal generators of the Lie group of E_2, in the differential form.

A matrix representation of the Euclidean group element can be obtained by associating with each point (x, y) in the plane a three dimensional vector (x, y, 1). Under the transformation

$T(\bar{a})\ R(\theta)$ the point (x, y, 1) becomes (x′ , y′, 1),

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad (2.2.8)$$

The matrix of transformations denoted by M (θ, a, b) is

$$M(\theta, a, b) = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$$

The Lie algebra corresponding to the Lie group of E_2 can be obtained by calculating the derivatives of M (θ, *a*, b) with respect to the three parameters θ, *a*, b around the identity . The infinitesimal matrix generators of the algebra are

$$L_a = \frac{\partial M}{\partial a}(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(2.2.9)$$

$$L_b = \frac{\partial M}{\partial b}(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_\theta = \frac{\partial M}{\partial \theta}(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.2.9)$$

The commutation relations are

$$[L_a, L_b] = 0,\ [L_a, L_\theta] = -L_b$$

$$[L_b, L_\theta] = -L_a \qquad (2.2.10)$$

It is to be noted that the general group element has been represented as the product of elements from two of its subgroups, which are the rotation group and the translation group, with $R(\theta)$ and $T(\bar{a})$ as the operators. These do not commute. In the product operation $R(\theta)\ T(\bar{a})$, $T(\bar{a})$ translates the origin into the point with coordinates (a,b), while the second operation $R(\theta)$ rotates this point (a,b) to the point with coordinates

$(a \cos\theta - b \sin\theta,\ a \sin\theta + b \cos\theta)$

This is equivalent to the transformation

$T(\tilde{\theta}_{\bar{a}})\ R(\theta)$

where $\tilde{\theta}_{\bar{a}}$ denotes a vector $\bar{a}$ rotated by $\tilde{\theta}$. Thus,

$$R(\theta)\ T(\bar{a}) = T(\tilde{\theta}_{\bar{a}})\ R(\theta) \qquad (2.2.11)$$

While the subgroup of translations is invariant , but that of rotations is not . The group, however, can be written as the product of a normal subgroup and a subgroup NH. Such that

$H \cap N = \{ e\}$

In other words , the group is a semi direct product group. We also note that the product of two group elements $T(\bar{a})\, R(\theta)$ and $T(\bar{a}')$ $R(\theta')$ is

$$T(\bar{a})\, R(\theta)\, T(\bar{a}')\, R(\theta') = T(\bar{a})\ T(\tilde{\theta}_{\bar{a}'})\, R(\theta)\, R(\theta')$$

$$= T(\bar{a}+\tilde{\theta}_{\bar{a}'})\, R(\theta+\theta') \qquad (2.2.12)$$

And, if R and S are arbitrary rotations, and $T(\bar{a})$ and $T(\bar{b})$ arbitrary translations, we have

$$R\, T(\bar{a}) = T(R_{\bar{a}})\, R$$

$$T(\bar{a})\, RT(\bar{b}) = T(\bar{a}+R_{\bar{b}})\, RS$$

where $R_{\bar{a}}$ denotes the vector $\bar{a}$ rotated by R .

We now discuss Frobenius method of induced representation (Frobenius, 1896 - 1899 [48]; Schur 1905 [99]; Burnside, 1911 [8]). The method is amenable to E_2. It provides that the representation D of a group G with a subgroup N also provides a representation of the subgroup N of G. The representation of N may be reducible , however, even if that of G , which D is irreducible . This is the case, since a subgroup may be invariant under the operators D(n), $n \in N$, but not under all the D(g), $g \in G$.

We now discuss the application of Frobenius method of induced representation to the $T(\bar{a})$ subgroup of E_2, which is normal, noting that this subgroup is also Abelian , so that the invariant subspace is one-dimensional. The irreducible representations of the translation subgroup $T(\bar{a})$ are of the form

$\exp(i\bar{p}.\bar{a})$

where $\bar{a}$ is the translation vector and $\bar{p}$ is an arbitrary vector of the space that labels the representation . If $D(\bar{a}, R) \equiv D(\bar{a}, \theta)$ is the total representation of E_2, then $D(\bar{a}, I)$ is reduced . It is expected that there are vectors ψ in the representation space H that satisfy

$$D(\bar{a}, I)\ \psi = e^{i\bar{p}\ \bar{a}}\psi \qquad (2.2.13)$$

Consider a vector $f(\bar{p})$ selected from a vector space Hp. Then all the vectors ψ of Hp are transformed by $D(\bar{a}, I)$ according to the equation $D(\bar{a}, I)\ \psi = e^{i\bar{p}\ \bar{a}}\psi$ So that (2.2.14)

$f(p) \in Hp$, we get

$$D(\bar{a}, I)\ f(\bar{p}) = e^{i\bar{p}\ \bar{a}} f(\bar{p}) \qquad (2.2.15)$$

We assume that $|\bar{p}|$ is fixed , so that f is a function of the direction ϕ of $\bar{p}$ only, where $0 \leq \phi \leq 2\pi$. The angle θ is the polar angle of $\bar{p}$. From $D(\bar{a}, I)\ f(\bar{p}) = e^{i\bar{p}\ \bar{a}} f(\bar{p})$, we see that $D(\bar{a}, I)$ is a local operator that multiplies the value of f at each point $\bar{p}$ by

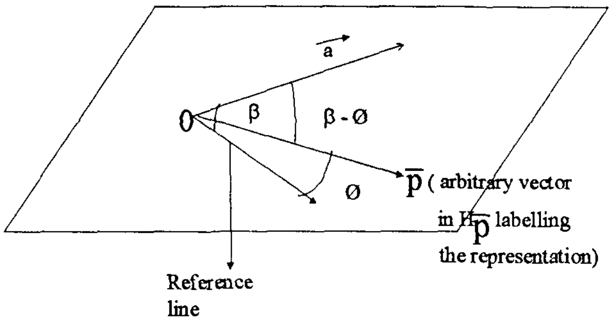
$\exp(i\bar{p}.\bar{a})$. Now,

$$D(\bar{a}, 0)\ f(\phi) = e^{ip|\bar{a}|}\cos(\beta - \phi)\ f(\phi) = e^{ipr\cos(\beta-\phi)}\ f(\phi) \qquad (2.2.16)$$

where $(\beta-\phi)$ is the angle between $\bar{p}$ and $\bar{a}$, with $|\bar{a}| = r$ [fig (1)] and $|\bar{p}| = p$.

The angle β is that made by the line of action of $\bar{a}$ with an arbitrary vector $\bar{p}$ in $H\bar{p}$, which labels the representation. The number p is an arbitrary positive number and is the index of the representation . Also,

$$D(0, \theta)\ f(\phi) = f(\phi - \theta)$$



(Fig. 1)

2.3 COMPLETE MATRIX REPRESENTATION OF E_2

The complete representation of the element $T(\bar{a})\ R(\theta)$ of E_2, for $p \neq 0$, is given by

$$D(\bar{a}, \theta)\ f(\phi) = [\ D(\bar{a},0)\ D(0,\theta)\ f]\ \phi$$

$$= D(\bar{a},0)\ \{D(0,\theta)\}\ f(\phi)$$

$$= \exp[ip|\bar{a}|\cos(\beta-\phi)]D(0,\theta)f(\phi)$$

$$=\exp[ipr\cos(\beta-\phi+\theta)f(\phi-\theta) \qquad (2.3.1)$$

There exists in a representation space a complete set of functions $f_{n\alpha}(\phi)$ satisfying the relation

$$D(0, \theta)\, f_{n\alpha}(\phi) = e^{-in\theta} f_{n\alpha}(\phi) \qquad (2.3.2)$$

where the index α is there to indicate that there can be more than one function satisfying the above equation . If there is only one function of f, we get

$$D(0, \theta)\, f(\phi) = f(\phi-\theta)$$

Putting $\phi = 0$, we get

$$f_n(-\theta) = e^{-in\theta} f_{n\alpha}(0) \qquad (2.3.3)$$

We can drop the index α, to get

$$f_n(-\theta) = e^{-in\theta} f_n(0)$$

Let $\theta \rightarrow -\theta$, and we obtain

$$f_n(\theta) = e^{in\theta} f_n(0)$$

We can replace θ by ϕ, to get

$$f_n(\phi) = e^{in\phi} f_n(0) \qquad (2.3.4)$$

We choose as a normalization

$$f_n(0) = i^{-n} (2\pi)^{-1/2}$$

Now that the f_n are normalized, the representation they define is unitary. Then,

$$f_n(\phi)=\frac{i^{-n}}{(2\pi)^{1/2}}e^{in\phi} \tag{2.3.5}$$

The matrix elements of the translation generators can be readily calculated from the relation

$$D(\bar{a},0)\ f_n = \sum_m \Delta_p(\bar{a},0)_{mn} f_m \tag{2.3.6}$$

where the L.H.S is an operator $D(\bar{a},0)$ of G on f_n, and

$\Delta_p(\bar{a},0)_{mn}$ is the matrix representing the operator $D(\bar{a},0)$, which is the translation operator of the translation subgroup of E_2. Recall

$$D(\bar{a},\theta)\ f_n(\phi) = \exp[\ irp\ \cos(\beta-\phi+\theta)]\ f_n\ (\phi-\theta) \tag{2.3.7}$$

and $D(\bar{a},\ 0)\ f_n\ (\phi) = \exp[\ irp\ \cos(\beta-\phi)]\ f_n(\phi)$ (2.3.8)

with $f_n\ (\phi)= \frac{i^{-n}}{(2\pi)^{1/2}}e^{in\phi}$

Hence

$$D(\bar{a},\ 0)\ f_n(\phi)= \exp[\ irp\ \cos(\beta-\phi)]\ \frac{i^{-n}}{(2\pi)^{1/2}}e^{in\phi}$$

$$= \frac{i^{-n}\ \exp[irp\ \cos(\beta-\phi)]\ e^{in\phi}}{(2\pi)^{1/2}} \tag{2.3.9}$$

Without any loss of generality, we can drop $1/(2\pi)^{1/2}$ factor to obtain,
$D(\bar{a},\ 0)\ f_n = \exp[irp\ \cos(\beta-\phi)]\ i^{n}\ \exp(in\phi)$

ie, $D(\bar{a}, 0)\, f_n = \sum_m \Delta_p(\bar{a},0)_{mn} f_m$

$$= \exp\,[irp\,\cos(\beta - \phi)]\; i^{-n}\; \exp(in\phi) \qquad (2.3.10)$$

Using $f_n\,(\phi) = i^{-n}\, e^{in\phi}$[on dropping the $1/(2\pi)^{1/2}$ factor], to obtain

$$f_m\,(\phi) = i^{-m}\, e^{im\phi}$$

so that

$$\sum_m \Delta(\bar{a},0)_{mn} f_m = \sum_m \Delta(\bar{a},0)_{mn} i^{-m} e^{im\phi}$$

ie $\qquad i^{-n}\,\exp(in\phi)\,\exp\,[ipr\,\cos(\beta-\phi)] = \sum_m \Delta(\bar{a},0)_{mn} f_m$

$$= \sum_m \Delta(\bar{a},0)_{mn} i^{-m} \exp(im\phi) \qquad (2.3.11)$$

This shows that $\Delta(\bar{a},0)_{mn}$ is the coefficient of $i^{-m}\, e^{im\phi}$ in the Fourier expansion of

$i^{-n}\,\exp\,[ipr\,\cos(\beta - \phi)]\,\exp(in\phi)$, in which case

$$\Delta(\bar{a},0)_{mn} = \frac{i^{m-n}}{2\pi} \int_0^{2\pi} \exp[irp\cos(\beta-\phi)]\exp[i(n-m)\phi]d\phi \qquad (2.3.12)$$

We integrate the R.H.S. by changing the variable of integration

$$\xi = \beta - \phi - \pi/2$$

so that $d\xi = -\,d\phi$, and so

$$\Delta_p(\bar{a},0)_{mn} = (-1)^{m-n} \exp[i(n-m)\beta]\; J_{m-n}(pr) \qquad (2.3.13)$$

whereby one identifies

$$J_{m-n}(pr) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-irp\sin\xi)\exp[i(m-n)\xi]d\xi \tag{2.3.14}$$

This is a familiar integral representation of $J_{m-n}(pr)$. It follows that

$$J_m(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-ix\sin\xi)\exp(im\xi)d\xi \tag{2.3.15}$$

which is a familiar integral representation of the Bessel function of the first kind of integral order m.

We next obtain the complete matrix representation $\Delta_p\ (\bar{a},\theta)_{mn}$ of $D(\bar{a},\theta)$.

We recall that

$$D(0,\ \theta)\ f_n(\phi) = e^{-in\theta}\ f_n\ (\phi)$$

$$= e^{in\phi}\ i^{-n}\ e^{-in\theta}$$

$$= e^{in(\phi-\theta)}\ i^{-n}$$

Then

$$\sum_m \Delta(0,\theta)_{mn} f_n(\phi) = \sum_m \Delta(0,\theta)_{mn} e^{-m} e^{im\phi}$$

$$= i^{-n}\ e^{in(\phi-\theta)} \tag{2.3.16}$$

showing that $\Delta(0,\ \theta)_{mn}$ is the coefficient of $e^{im\phi}\ i^{-m}$ in the Fourier expansion of $e^{im(\phi-\theta)}$, in which case

$$\Delta(0,\ \theta)_{mn} = \frac{i^{m-n}}{2\pi} \int_0^{2\pi} \exp[in(\phi-\theta)]d\phi$$

$$= \frac{i^{m-n}}{2\pi} \exp(-in\theta) \qquad (2.3.17)$$

The complete representation $\Delta_p(\bar{a},\theta)_{mn}$ is now given , with the matrix elements, as

$$\Delta_p(\bar{a},\theta)_{mn} = [\Delta_p(\bar{a},0)\ \Delta_p\ (0,\theta)]_{mn}$$

$$= (-1)^{m-n} \exp(-im\beta) \exp(in\beta) J_{m-n}(pr) \exp(-in\theta)$$

$$= (-1)^{m-n}\exp(-im\beta)J_{m-n}(pr)\exp[in(\beta-\theta)] \qquad (2.3.18)$$

where (r, β) are the polar coordinates of $\bar{a}$, such that $|\bar{a}|=r$, arg $(\bar{a})=\beta$.

We now discuss the method of obtaining the power series of a Bessel function of the first kind, of integral order, $J_m(a)$, Recall

$$\Delta_p\ (\bar{a},\theta)_{mn} = (-1)^{m-n} \exp(-im\beta)\ J_{m-n}(pr)\ \exp[in(\beta-\theta)]$$

The index p which labels the representation is irrelevant to the development of a power series for J_m and its other properties. We therefore take p as 1. Consider the transformations by the vector a, parallel to the x-axis of the plane , in order to obtain the power series for J_m. In this case the equation

$$D(\bar{a},0)\ f_n = \sum_m \Delta_p(\bar{a},0)_{mn} f_m$$

becomes

$$D(\bar{a},0)\psi_n = \sum_m (-1)^{m-n}\exp[i(n-m)\beta]J_{m-n}(pr)\psi_m \qquad (2.3.19)$$

(evaluated at p = 1 as suggested), here $|\bar{a}| = r = a$. The polar angle of $\bar{a}$ is β, and since $\bar{a}$ is parallel to the x-axis of the plane , we have that β =0 . Hence

$$D(\bar{a},0)\ \psi_n = \sum_m (-1)^{m-n} J_{m-n}(a)\psi_m \tag{2.3.20}$$

But a translation of a distance a in the x- direction can be written operationally as

$$D(\bar{a},0) = e^{aL_a} \text{ (by exponentiation)} \tag{2.3.21}$$

where L_a is the corresponding generator of the Lie algebra corresponding to the subgroup of translation by the vector $\bar{a}$. Since the generators are not linearly independent, it is impossible to construct representations of the Euclidean group E_2 from the commutation relations

$$[L_a, L_b]= 0,\ [L_a, L_\theta]= -L_b,\ [L_b, L_\theta]= L_a$$

The method of constructing irreducible representations for E_2 is to calculate, first, the irreducible representations of the Lie algebra. We need three independent generators other than L_a, L_b, L_θ. One is interested in unitary group representations. We try to find skew-Hermitian representations of the algebra. Consider the infinitesimal operators

P^+ , P^- defined by

$$P^+ = L_a + i\ L_b;\quad P^- = (-P^+)^+ = L_a - iL_b \tag{2.3.22}$$

Recall that commutation relations of L_a, L_b, L_θ namely

$$[L_\theta, L_a] = - [L_a, L_\theta] = -L_b$$

Now,

$$[L_\theta, P^+] = -L_b - iL_a = -(L_b + iL_a)$$

$$= -i\ (L_a + iL_b) = -iP^+$$

$$[L_\theta, P^-] = iP^-$$

The Casimir invariant operator is

$$P^2 = L_a^2 + L_b^2 = P^+P^- = P^-\ P^+ \qquad (2.3.23)$$

with P^2 commuting with L_a , L_b and L_θ. Therefore P^2 is, if the representation is irreducible, a non positive real constant , which we denote by $-p^2$, i.e.,

$$P^2 = -p^2,\ p \neq 0 \qquad (2.3.24)$$

From $(L_a + iL_b) = P^+$, $(L_a - iL_b) = P^-$, we obtain

$$L_a = (P^+ + P^-)/2,\ L_b = (P^+ - P^-)/2i$$

Then $D(\bar{a},0) = \exp(aL_a)$

$$= \exp(a\ (P^+ + P^-)/2)$$

$$= \sum_{s=0}^{\infty} \frac{1}{s!}(P^+ + P^-)^s \left(\frac{a}{2}\right)^s$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{s!(P^+)^r(P^-)^{s-r}}{s!r!(s-r)!}\left(\frac{a}{2}\right)^s$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(P^+)^r(P^-)^{s-r}}{r!(s-r)!}\left(\frac{a}{2}\right)^s \qquad (2.3.25)$$

It is necessary at this stage to invoke group properties in order to ensure that the representation of the algebra should correspond to a representation of E_2. The property to be invoked is that E_2 has a compact subgroup which is the rotation group. The chosen parameterization, in which $R(2\pi) = I$, requires that the eigen value of L_θ be in , where n is an integer , in order that $e^{aL_a} = I$. We take ψ_n as a normalized eigen vector of L_θ satisfying

$$L_\theta \psi_n = -in\psi_n \qquad (2.3.26)$$

There are two different cases : The first case is that for which $p^2 = 0$, implying

$$P^+ P^- \psi_n = P^-P^+ \psi_n = 0$$

So that

$$(\psi_n, P^+ P^- \psi_n) = - | P^- \psi_n|^2 = 0$$

in which case $P^- \psi_n = 0$

Similarly, $P^+\psi_n = 0$

For this case the complete representation is defined by ψ_n, which is one dimensional and is of the rotation subgroup. This representation is of little interest.

The second case is that in which $p^2>0$. For all u in the domain of P^+ and P^-, we have that P^+u and $P^- u$ are non zero, otherwise $p^2 = 0$. We consider $P^+\psi_n$ with this

$$[L_\theta, P^+] = -i P^+$$

gives $L_\theta (P^+\psi_n) = P^+ L_\theta \psi_n - i P^+\psi_n$ using $L_\theta \psi_n = -in \psi_n$, this becomes

$$L_\theta (P^+\psi_n) = P^+ (-in \psi_n) - iP^+\psi_n$$

$$= -i(n+1)P^+\psi_n$$

indicating that $P^+\psi_n$ is an eigenvector of L_θ, corresponding to the eigenvalue $-i(n+1)$.

Similarly, $P^-\psi_n$ is an eigenvector of L_θ corresponding to the eigenvalue $-i(n-1)$. The non normalized eigenvectors $P^+\psi_n$, $P^- \psi_n$ satisfy

$$| P^+\psi_n|^2 = (\psi_n, P^-P^+\psi_n) = p^2$$

and $| P^-\psi_n|2 = p^2$

We next define the normalized eigenvectors of L_θ by

$$\psi_{n+1} = (-P^+/p) \psi_n \quad , \psi_{n-1}= (P^-/p) \psi_n$$

The phases of ψ_{n+1}, ψ_{n-1} can be fixed arbitrarily; these have chosen so that the representations obtained will agree with the equation of the complete representation

$$\Delta_p\ (\bar{a},\theta)_{mn} = (-1)^{m-n} \exp(-im\beta)\ J_{m-n}(pr)\ \exp[in(\beta-\theta)]$$

We inductively define

$$\psi_{n+m} = (-P^+/p)^m \psi_n \qquad , \ \psi_{n-m} = (P^-/p)^m \psi_n$$

which are again eigen vectors of L_θ corresponding to the eigen values -i(n+m) and -i(n-m) respectively. We note that

$$(-P^+/p\ )\ \psi_{m+n} = \psi_{m+n+1}$$

$$(P^-/p)\ \psi_{n-m} = \psi_{n-m-1}$$

for m>0 and $\quad (P^-/p\ )\ \psi_{m+n} = \psi_{m+n-1}$

$$(P+/p\ )\ \psi_{n-m} = \psi_{n-m+1}$$

for $m \geq 1$. One concludes that the eigen vectors ψ_m, $m=n,\ n\pm1$, provide a complete definition of the Lie algebra by

$$L_\theta\ \psi_m = -im\psi_m$$

$$P^+\ \psi_m = -p\psi_{m+1}$$

$$P^-\ \psi_m = p\psi_{m-1},\ m=n,\ n\pm1$$

This construction is independent of the choice of the eigenvector ψ_n in

$L_\theta\ \psi_n = -in\psi_n$.

for if another eigen vector of L_θ had been chosen , the same sequence of eigenvectors would have been found . The eigenvectors of L_θ are also non degenerate. Degeneracy would lead to reducibility in representation. The representations are necessarily infinite dimensional, since the eigenvectors $\psi_{n\pm m}$ defined by

$$\psi_{m+n} = (-P^+/p)^m \psi_n$$

$$\psi_{n-m} = (P^-/p)^m \psi_n$$

cannot vanish for any value of m and are necessarily linearly independent. We now return to (2.3.25). Recall equation (2.3.6) and replace f_n by ψ_n, to obtain

$$D(\bar{a},0)\psi_n = \sum_m (-1)^{m-n} J_{m-n}(a) \Psi_m$$

in the special case of n = 0, we get

$$D(\bar{a},0)\psi_0 = \sum_m (-1)^m J_m(a) \Psi_m$$

Compare this to (2.3.25) to obtain

$$D(\bar{a},0) = \sum_m (-1)^m J_m(a) \Psi_m$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(P^+)^r (P^-)^{s-r}}{r!(s-r)!} \left(\frac{a}{2}\right)^s$$

We recall

$(-P^+/p)\ \psi_{m+n} = \psi_{m+n+1};\ (P^-/p\ )\ \psi_{n+m} = \psi_{n+m-1}$

with n=0, i.e. ,

$(-P^+/p\ )\ \psi_m = \psi_{m+1};\ (P^-/p)\ \psi_m = \psi_{m-1}$

or $P^+\psi_m = -p\ \psi_{m+1};\ P^-\psi_m = p\ \psi_{m-1}$

From these we obtain

$$(P^+)^r\ (P^-)^{s-r}\ \psi_0 = (-1)^r\ \psi_{2r-s}$$

We substitute this, to obtain

$$\sum_m (-1)^m J_m(a)\Psi_m = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(s-r)!}\left(\frac{a}{2}\right)^s \Psi_{2r-s}$$

$$= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(s)!}\left(\frac{a}{2}\right)^{r+s} \Psi_{2r-(r+s)}$$

$$= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(s)!}\left(\frac{a}{2}\right)^{r+s} \Psi_{r-s} \tag{2.3.27}$$

The series for J_m is obtained by equating the coefficients of ψ_n on both sides of the identity . Two cases arise, namely

$m \geq 0$, and $m<0$

For $m \geq 0$, the terms on the R.H.S. for which r-s=m or r=s+m give the desired result, which is the following :

$$J_m(a) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+m)!}\left(\frac{a}{2}\right)^{2s+m} \tag{2.3.28}$$

For $m<0$, the terms for which $s=r-m$ give the result

$$J_m(a)=(-1)^m\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r-m)!}\left(\frac{a}{2}\right)^{2r-m}$$

$$= (-1)^m J_{-m}(a) \qquad (2.3.29)$$

which shows that J_m and J_{-m} for integral m , are linearly dependent and cannot be combined linearly to give a general solution of the Bessel differential equation of the first kind of order m. These power series are convergent for all values of the argument.

We next look at the general addition theorem and its implications for the Bessel function, which are recurrence relations . The most general addition theorem for E_2 is

$$\Delta\ (\bar{a}+\tilde{\theta}_{\bar{a}'}, \theta+\theta')_{mn} = \Sigma\ \Delta(\bar{a},\theta)_{mp}\ \Delta\ (\bar{a}',\ \theta')_{pn}$$

Without any loss of generality we can put $\theta=\theta'=0$. Again we assume $\bar{a}$ parallel to the x- axis , since this can be achieved by a simultaneous rotation of $\bar{a}$ and $\bar{a}'$. We conveniently express the addition theorem in cartesian coordinates. We write

$\exp(-i\beta') = \cos\beta' - i\sin\beta'$, with $\cos\beta' = a'/r'$, and $\sin\beta' = b'/r'$, so that

$\cos\beta' - i\sin\beta' = (a'-ib')/r'$

Now, $\Delta(\bar{a}',\ 0)_{pn} = (-1)^{p-n}\ [(a'-ib')/r']^{p-n}\ J_{p-n}(r')$

with n=0. Since $\bar{a}$ is parallel to the x-axis, then $\beta=0$; $\bar{a}+\bar{a}'$ has components r+a, b', with $|\bar{a}|$=r. We note that

$$\Delta(\bar{a},\ 0)_{mp}=(-1)^{m-p}\ J_{m-p}(r)$$

Without any loss of generality, in the special case of m=0, we obtain

$$((r+a'-ib')/R)^m\ J_m(R)=\ \sum_p \left(\frac{a'-ib'}{r'}\right)^p J_{m-p}(r)J_p(r') \qquad (2.3.30)$$

with $R^2 = (r+a')^2+b'^2$;

$r'^2=a'^2+b'^2$

and summation is over all integral n values of p. In polar coordinates, we parametrize:

$(r+a')/R = \cos B \quad , \quad b'/R = \sin B$

by De Moivre's theorem we obtain

$$\left(\frac{r+a'-ib'}{R}\right)^m = (\cos B - i \sin B)^m=e^{-imB}$$

so that

$$\left(\frac{r+a'-ib'}{R}\right)^m\ J_m(R)=\sum_p \left(\frac{a'-ib'}{r'}\right)^p J_{m-p}(r)J_p(r') \qquad (\ **\ )$$

becomes

$$e^{-imB}J_m(R)=\ \sum_p \exp(-ip\beta')J_{m-p}(r)J_p(r')$$

with $R^2=(r+a')^2+b'^2=r^2+2ra'+a'^2+b'^2$

$$= r^2+2rr' \cos\beta'+a'^2 + b'^2$$

$$= r^2 + 2r\, r' \cos \beta'+r'^2$$

In the special case of b'=0, corresponding to the product of translations along the x- axis, equation (**) reduces to

$$J_m(a+a')= \sum_p J_{m-p}(a)J_p(a') \qquad (2.3.31)$$

Similarly, if a'=0, the identity is, on replacing b' with b,

$$\left(\frac{a-ib}{(a^2+b^2)^{1/2}}\right)^m J_m((a^2+b^2)^{1/2})= \sum_p (-1)^p J_{m-p}(a)J_p(b) \qquad (2.3.32)$$

Equations (2.3.31) and (2.3.32) lead to well known recurrence relations for the Bessel functions, which we obtain as follows :

Differentiate equation (2.3.31) with respect to a' and evaluate at a'=0, to obtain

$$J'_m(a) = J_{m-1}(a)-J_{m+1}(a) \qquad (2.3.33)$$

Similarly, equation (2.3.32) is differentiated with respect to b and evaluated at b=0, to obtain

$$(2m/a)\, J_m(a) = J_{m-1}(a) + J_{m+1}(a) \qquad (2.3.34)$$

2.4 PARTIAL DIFFERENTIAL EQUATION OF HELMHOLTZ FOR $\Delta(a)\equiv\Delta(r,\beta)$ FOR EUCLIDEAN GROUP E_2 FOR THE PLANE :

We consider the identity

$$\Delta(\bar{a},0)\Delta(\bar{a}',0) = \Delta(\bar{a}'+\bar{a}',0) \tag{2.4.1}$$

where $\Delta(\bar{a},0)_{mn}=(-1)^{m-n}\ e^{i(n-m)\beta}\ J_{m-n}(pr)$

for $\beta=0$, $p=1$

$$\Delta(\bar{a},0)_{mn}=(-1)^{m-n}\ J_{m-n}(r) \tag{2.4.2}$$

and

$$\Delta(\bar{a}',0)_{mn}=(-1)^{m-n}\left(\frac{a'-ib'}{r'}\right)^{m-n} J_{m-n}(r') \tag{2.4.3}$$

The Helmholtz equation arises from the above identity, i.e. from equation (2.4.1). Differentiate equation (2.4.1) with respect to a' and b' and evaluate the results at $a'=0$, we get

$$\frac{\partial\Delta}{\partial a}\ (\bar{a})= \Delta(\bar{a})\ L_a \tag{2.4.4}$$

$$\frac{\partial\Delta}{\partial b}\ (\bar{b})= \Delta(\bar{a})\ L_b \tag{2.4.5}$$

Alternatively, from $D(\bar{a},0) = e^{aL_a}$(exponentiation) so that $\Delta\ (\bar{a},\ 0)_{mn}= e^{aL_a}$, we obtain

$$\frac{\partial\Delta}{\partial a} = L_a e^{aL_a} = \Delta L_a$$

Similarly,

$$\frac{\partial\Delta}{\partial b} = \Delta L_b$$

Differentiating again with respect to a and b respectively, and apply the Casimir operator

$L_a^2+L_b^2 = -p^2=-1$, without any loss of generality, as follows

$$\frac{\partial^2 \Delta}{\partial a^2} = \frac{\partial}{\partial a} (\Delta(\bar{a}))L_a$$

$$\frac{\partial^2 \Delta}{\partial b^2} = \frac{\partial}{\partial a} (\Delta(\bar{a}))L_b$$

we get

$$\frac{\partial^2 \Delta}{\partial a^2}+\frac{\partial^2 \Delta}{\partial b^2} = (\Delta(\bar{a}))L_a^2+(\Delta(\bar{a}))L_b^2$$

$$= (L_a^2 + L_b^2)\ \Delta(\bar{a})$$

$$= -p^2\ \Delta(\bar{a}) = -\Delta(\bar{a})$$

hence

$$\frac{\partial^2 \Delta}{\partial a^2}(\bar{a})+\frac{\partial^2 \Delta}{\partial b^2}(\bar{a})+\Delta(\bar{a})=0 \qquad (2.4.6)$$

which is the two dimensional Helmholtz equation satisfied by each matrix element of the representation of the translation operator D ($\bar{a}$, 0) , with the matrix element of the representation as $\Delta(a,0)_{mn}$. In polar coordinates (r,β), we obtain

$$\Delta_{rr}+1/r\Delta_r+1/r^2\Delta_{\beta\beta}+\Delta(r,\beta)=0 \qquad (2.4.7)$$

In terms of the Laplace operator

$$\nabla^2\Delta(r,\beta)+\Delta(r,\beta) = 0 \qquad (2.4.8)$$

From the Helmholtz differential equation we deduce , in a straight forward manner, the Bessel differential equation of the first kind of integral order m, $J_m(r)$, with $|\bar{a}|=r$;

We recall

$$\nabla^2\Delta(r,\beta)+\Delta(r,\beta) = 0$$

and

$$\Delta_p(\bar{a},0)_{mn}=(-1)^{m-n} \exp[i\ (n-m)\beta]\ J_{m-n}(pr)$$

with p=1, as specialized earlier, putting $\beta=0$, $|\bar{a}|=r$, we obtain

$$J_m''(r) + 1/r\ J_m'(r) + (1-m^2/r^2)J_m(r) = 0 \qquad (2.4.9)$$

which is the Bessel differential equation of the first kind of order m in $J_m(r)$, with m an integer.

We can also easily deduce some recurrence relations by combining

$$2\ J_m'(a) = J_{m-1}(a) - J_{m+1}(a)$$

and $(2m/a)\ J_m(a) = J_{m-1}(a) + J_{m+1}(a)$

to obtain

$$J_{m-1}(r) = J_m'(r) + m/r\ J_m(r) \qquad (2.4.10)$$

with

$|\bar{a}|=a=r$ and $J_{m+1}(r) = -J_m'(r) + m/r\ J_m(r)$ (2.4.11)

Equations (2.4.10) and (2.4.11) are well known recurrence relations for the Bessel functions .

By adding equations (2.4.10) and (2.4.11), we obtain another recurrence relation

$$J_{m-1}(r) + J_{m+1}(r) = (2m/r)\, J_m(r) \qquad (2.4.12)$$

The two relations (2.4.10) and (2.4.11) provide the factorization of the Bessel equation .

Finally, we work out the generating function for the Bessel equation from the matrix representation $\Delta_p(\bar{a},0)_{mn}$ of $D(\bar{a},0)$. We recall that $\Delta_p(\bar{a},0)_{mn}$ have been defined as the coefficients of $i^{-m}e^{im\phi}$ in the Fourier expansion of

$\exp[\, ir\cos(\beta-\phi)]\, i^{-n}\exp(\, in\phi)$. In the special case of $\beta=\pi/2$, for which $\Delta(\bar{a},0)_{mn}=i^{m-n}\, J_{m-n}(r)$, this property gives

$$i^{-n}\, e^{ir\sin\phi}\, e^{in\phi} = \sum_m i^{m-n} j_{m-n}(r) i^{-m} e^{im\phi} \qquad (2.4.13)$$

Substitute $z = e^{i\phi}$, so that

$\mathrm{Sin}\ \phi=(z-z^{-1})/2i, \qquad \cos\phi = (z+z^{-1})/2$

to get

$$i^{-n}\exp[ir(z-z^{-1})/2i]\ \exp(in\phi)=\sum_m i^{m-n} j_{m-n}(r) i^{-m} e^{im\phi}$$

or,

$$i^{-n} \exp[ir(z-z^{-1})/2i] \exp(in\phi) = i^{-n} \sum_m i^{-m} j_{m-n}(r) i^{-m} e^{im\phi}$$

or,

$$\exp[ir (z-z^{-1})/2i] \exp(in\phi) = \sum_m j_{m-n}(r) i^{-m} e^{im\phi}$$

As a special case putting n = 0 , to get

$$\exp[ir (z - z^{-1})/2i] = \sum_m j_m(r) z^m \tag{2.4.14}$$

This result has been demonstrated for $|z| = 1$. The series converges for other values of z and can be extended to these values by analytic continuation . The L.H.S of equation (2.4.14) is called the generating function of the Bessel function $J_m(r)$. The result of this section can be extended to complex values of the arguments of the Bessel functions and for complex transformations vectors $\bar{a}$, $\bar{a}'$.

CHAPTER - 3

LIE ALGEBRA AND SPECIAL FUNCTIONS

3.1 INTRODUCTION :

Many of the important classical differential equations are related with Lie theory . We discuss some properties of special functions, Rodrigues - type formulae and differential equations using some operators defined on a Lie algebra. Due to noticeable importance of Lie algbraic approach for applications, it deserves some interest to get, in addition to the results obtained by Roger Howe [54] , Radulescu [91] and Chiccoli et al [22] further mathematical relations for these functions. The analytic methodology developed in the study can easily be adopted to the study of some other special functions of mathematical physics.

Let V be a finite dimensional vector space. Let EndV denote the algebra of linear maps from V to itself, and let GL(V) denote the group of invertible linear maps from V to itself. The usual name for GL(V) is the general linear group of V. If $V= R^n$, then EndV= $M_n(R)$, the n x n matrices, and GL(V)= GL(n, R), the matrices are non vanishing determinants.

Let $V = C^{\infty}(R)$. We define the operators A, $B \in$EndV by

$$(Af)x = a(x)\frac{df}{dx}+b(x)f(x), \qquad (Bf)x = c(x)\frac{df}{dx}+e(x)f(x) \tag{3.1.1}$$

for every $x \in R$, then,

$$[A, B]f = -c(x)\, b(x)\, f' + [\, a(x)\, e'(x) - e(x)\, b(x) - c(x)\, b'(x)]f \quad (3.1.2)$$

Define the sequence $(y_n)_n \subset V$ for every $n \geq 1$, and $L = BA$, then

$$Ly_n = BAy_n = c(x)\, a(x)\, y_n'' + [c(x)\, a'(x) + c(x)\, b(x) + e(x)\, a(x)]y'_n +$$

$$[c(x)\, b'(x) + c(x)\, b(x)]\, y_n \quad (3.1.3)$$

Let A, B and the identity operator I span a Lie algebra with commutation relation $[A, B] = AB - BA = I$.

We choose first to recall here the main theorem of Radulescu [91] (see also Howe [54]).

3.2 MAIN THEOREM :

Theorem 3.2.1 Let $A, B \in \mathrm{End}V$ be such that $[A, B] = I$. We define the sequence $(y_n)_n \subset V$ as follows: $Ay_0 = 0$ and $y_n = By_{n-1}$, for any $n \geq 1$. Then y_n is an eigen vector of eigen value n for BA, for every $n \geq 1$.

Proof :

Given that $y_n = By_{n-1}$

First we will show that $Ay_n = ny_{n-1}$ for every $n \geq 1$ (3.2.1)

Since $Iy_0 = [A, B]\, y_0 = ABy_0 - BAy_0$, $\quad Ay_0 = 0$ and $y_1 = By_0$

$$= ABy_0 = Ay_1 = y_0$$

Thus (3.2.1) is true for n=1

Let us suppose that the result is true for any positive integer n, i.e. $Ay_n = ny_{n-1}$

We may write equivalently ,

$[A, B]\, y_n = y_n$

$ABy_n - BAy_n = y_n$

$Ay_{n+1} - nBy_{n-1} = y_n$

$Ay_{n+1} - ny_n = y_n$

$Ay_{n+1} = (n+1)y_n$

Thus (3.2.1) is true for n+1. Now

$BAy_n = nBy_{n-1} = ny_n$

Hence y_n is an eigen vector of eigen value n for every $n \geq 1$.

Let $V = C^{\infty}(R)$. We define the operators $A, B \in \text{End}V$ by

$(Af)\, x = (1/2)\, f'(x)$, $(Bf)x = -f'(x) + 2x\, f(x)$, for every $x \in R$

We prove that these operators satisfy the commutation relation

$[A, B] = I$ Indeed,

$$(A(Bf))x - B((Af)x) = -(1/2)\, f''(x) + xf'(x) + f(x) + (1/2)f''(x) - xf'(x)$$

$$= f(x)$$

Next, we prove the following result

Proposition (3.2.1)

$$(B^n f)x = (-1)^n e^{x^2} (f(x)e^{-x^2})^{(n)}$$

From the definition of B, the above equality holds for n= 1.

Inductively, taking into account $B^{n+1}f= B(B^n f)$, it follows that

$$(B^{n+1}f)x= -(-1)^n (e^{x^2} (f(x)e^{-x^2})^{(n)})' +2x(-1)^n e^{x^2} (f(x)e^{-x^2})^{(n)}$$

$$=(-1)^{n+1}e^{x^2} (f(x)e^{-x^2})^{(n+1)}$$

Hence, by the method of mathematical induction we have

$$(B^n f)x = (-1)^n e^{x^2} (f(x)e^{-x^2})^{(n)}$$

which ends our proof.

The Hermite equation $y''- 2xy'+ 2ny=0$, where n is a positive integer, may be written

$-y'' +2xy' = 2ny$, or $By'= 2ny$ i.e. $BAy = ny$.

Setting $y_0=1$, we obtain $y_n= B^n(1)$. Therefore, defining $H_n(x)=y_n$, we deduce the Rodrigues- type formula

$H_n(x)= (-1)^n e^{x^2} (e^{-x^2})^{(n)}$, for every positive integer n.

In this manner we are able to deduce other properties of Hermite functions.

Proposition (3.2.2) :

The Hermite functions satisfy the following recursion relations:

i) $H_n'(x) = 2nH_{n-1}(x), \quad n \in N$

ii) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \; n \in N$

Proof :

i) Since $H_n(x) = y_n = B^n(1)$, we can use $Ay_n = n\, y_{n-1}$. From the definition of the operator A, it follows that

$$(1/2)\, y_n' = ny_{n-1} \text{ i.e. } H_n'(x) = 2n\, H_{n-1}(x)$$

ii) By the definition of B , we have

$$By_n + y_n' - 2xy_n = 0$$

But $y_n' = 2ny_{n-1}$. Thus,

$$By_n - 2xy_n + 2ny_{n-1} = 0$$

or $\quad y_{n+1} - 2xy_n + 2ny_{n-1} = 0$

i.e $H_{n+1}(x) - 2xH_n(x) + 2n\, H_{n-1}(x) = 0$

Theorem (3.2.2) :

Let A, B $\in$ EndV such that $[A, B] = I$. Define $S=(A-I)\, BA$, $T_n = (A-I)^n B^n$, for every $n \geq 1$. Then

(a) $\quad [(A-I)^n, B] = n(A-I)^{n-1}$

(b) $T_{n+1} = (T_1 + nI)T_n$

(c) $S(T_1 + nI) = (T_1 + nI)S + (S+nI) - (T_1 + nI)$

(d) If $y_0 \in V$ is an eigen vector of S with eigenvalue 0, then y_n is an eigenvector of eigenvalue -n for S, where $y_n = T_n\, y_0$, for every $n \geq 1$.

(e) If $y_n \in V$ is an eigen vector of eigen value -n for S, then $w_n = (T_1 + nI)\, y_n$ is an eigen vector for S, with eigen value -(n+1)

Proof :

(a) For n=1, the equality follows from the commutation relation $[A, B] = I$. Inductively, let us suppose that

$[(A-I)^n\ B] = n\,(A-I)^{n-1}$

It follows that

$(A-I)^n B - B(A-I)^n = n(A-I)^{n-1}$

or, $B(A-I)^n = (A-I)^n B - n(A-I)^{n-1}$

$$
\begin{aligned}
B(A-I)^{n+1} &= (A-I)^n B(A-I) - n(A-I)^n \\
&= (A-I)^n (BA-BI-nI) \\
&= (A-I)^n (AB-I-BI-nI) \\
&= (A-I)^n((A-I)B-(n+1)I) \\
&= (A-I)^{n+1} B-(n+1)(A-I)^n
\end{aligned}
$$

or, $B(A-I)^{n+1} = (A-I)^{n+1}B-(n+1)(A-I)^n$

or, $(A-I)^{n+1}B-B(A-I)^{n+1}=(n+1)(A-I)^n$

or, $[(A-I)^{n+1}, B]=(n+1)(A-I)^n$

This shows that the result is true for n+1 when it is true for n, therefore by the method of induction the result is true for any positive integer $n \geq 1$. i.e.

$$[(A-I)^n, B]= n(A-I)^{n-1}$$

(b) From (a) , we have

$[(A-I)^n, B]B^n = n(A-I)^{n-1} B^n$

$(A-I)[(A-I)^n, B] B^n=(A-I)n (A-I)^{n-1} B^n$

or, $(A-I)\{(A-I)^n B-B(A-I)^n\}B^n=n(A-I)^n B^n$

or, $(A-I)^{n+1} B^{n+1} - (A-I) B(A-I)^n B^n = n(A-I)^n B^n$

or, $(A-I)^{n+1} B^{n+1} = n(A-I)^n B^n +(A-I) B(A-I)^n B^n$

or, $T_{n+1} = nI\, T_n + T_1 T_n$

$\qquad = (nI+ T_1) T_n$

(c) The equality we have to prove is equivalent to

$$[S, T_1] = S- T_1$$

But

$$[S, T_1] = ST_1 - T_1S$$

$$= (A\text{-}I)\ BA\ (A\text{-}I)\ B\text{-}\ (A\text{-}I)\ B(A\text{-}I)BA$$

$$= (A\text{-}I)\ B(A^2\ B\text{-}AB\text{-}ABA\text{+}BA)$$

$$= (A\text{-}I)B(A[A,\ B]\text{-}\ [A,\ B])$$

$$= (A\text{-}I)B(A\text{-}I)$$

$$= (A\text{-}I)\ BA\text{-}(A\text{-}I)B$$

$$= S\text{-}T_1$$

(d) We will prove our assertion by induction. Since $Ay_0 = 0$, our assertion is true for n=0. Now suppose

$$Sy_n = -ny_n.\ \text{But}\quad y_n = T_n y_0$$

Thus $ST_n y_0 = -nT_n y_0$

For n=1, this equality is true. Now we find $ST_{n+1}y_0$.

$$ST_{n+1}y_0 = s(T_1 + nI)\ T_n y_0\ \text{[from (b)]}$$

and

$$ST_{n+1}y_0 = [(T_1\text{+}nI)\ S\text{+}(S\text{+}nI)\text{-}(T_1\text{+}nI)]T_n y$$

$$= (T_1\text{+}nI)\ ST_n y_0\ \text{+}(S\text{+}nI)\ T_n y_0\text{-}(T_1\text{+}nI)T_n y_0$$

$$= (T_1\text{+}nI)\ (n\text{-}T_n y_0)\ -\ T_{n+1}y_0$$

$$= -n\ T_{n+1}\ y_0\ \text{-}T_{n+1}y_0$$

$$S\, T_{n+1}y_0 = -(n+1)T_{n+1}y_0$$

Thus the result is true for (n+1) when it is true for n. Therefore, by the method of induction the result is true for any positive integer n i.e. y_n is an eigen vector of eigen value -n for S.

(e) From (c), it follows that

$$Sw_n = (T_1 + nI)\, Sy_n + (S+nI)\, y_n - w_n$$

But, $Sw_n = -n(T_1+nI)\, y_n - w_n$ [Since, $(S+nI)\, y_n = 0$]

$$= -(n+1)\, w_n$$

This w_n is an eigen vector for S, with eigen value -(n+1).

For $V = C^\infty(R)$, we define the operators A, B $\in$ EndV by

$(Af)\, x = f\,'(x)$ and

$(Bf)x = xf(x)$, for every $x \in R$ and $f \in C^\infty(R)$, A and B satisfy $[A,B] = I$

We consider now the Laguerre equation [92]

$$xy''+(1-x)y' + ny = 0,\ n \in N$$

But $(A-I)BAy = xy''+(1-x)y'$. Thus the Laguerre equation becomes

$$Sy = -ny$$

By theorem (3.2.2) , we conclude that if $y_0 \in C^\infty(R)$ and $Sy_0=0$, then $y_n = T_n\, y_0$ is a solution of the Laguerre equation. We choose $y_0=1$. It is clear that $Sy_0=0$

We shall prove now that

$$((A-I)^n f)x=e^x(f(x)e^{-x})^{(n)}, \text{ for every } n\in N \tag{3.2.2}$$

For n=1, this equality becomes

$f'(x)-f(x)=e^x(f(x)e^{-x})'$, which is trivial

Inductively, we suppose that relation (3.2.2) is true. Therefore,

$$\begin{aligned}(A-I)^{n+1} f)x &= ((A-I)(A-I)^n f)x \\ &= e^x ((A-I)^n f) xe^{-x})' \\ &= e^x (e^x (f(x) e^{-x})^{(n)} e^{-x})' \\ &= e^x (f(x) e^{-x})^{(n+1)}\end{aligned}$$

which proves the validity of relation (3.2.2)

Since $B^n(1)=x^n$, it follows that

$$\begin{aligned}y_n = T_n y_0 &= (A-I)^n B^n(1) \\ &= e^x (x^n e^{-x})^{(n)}\end{aligned}$$

Thus the Laguerre function

$L_n(x) = e^x (x^n e^{-x})^{(n)}$ is a solution of the Laguerre equation.

Proposition (3.2.3) :

The Laguerre functions satisfy the following resursion relations

i) $L'_{n+1}(x) - (n+1) L_n'(x) + (n+1) L_n(x) = 0$

ii) $L_{n+1}(x) + (x-2n-1) L_n(x) + n^2 L_{n-1}(x) = 0$, for every $n \in N$.

Proof:

i) Using the definition of A and $L_n(x)=T_n(1)$, our relation becomes

$$AT_{n+1}(1) - (n+1)AT_n(1)+(n+1) T_n(1)=0$$

From (b) of the theorem (3.2.2) the above equality becomes

$$AT_1T_n(1) - AT_n(1) +(n+1)T_n(1)=0$$

or $(AT_1-A+I)y_n = -ny_n$,

because $T_n(1) = y_n$, [by theorem 3.2.2(d)]

Thus by the same theorem, it suffices to prove that $AT_1- A+I =S$.

Indeed,

$$\begin{aligned} AT_1 - A+I-S &= A(A-I)B-A+I-(A-I)BA \\ &= A^2B-AB-A+I-(A-I)BA \\ &= A^2B-A-ABA \\ &= A(AB-BA)-A \\ &= A[A,B]-A=0 \end{aligned}$$

ii) By the definition of B, we have to prove that

$$T_{n+1}(1)+ BT_n(1)-(2n+1)T_n(1) + n^2T_{n-1}(1)=0$$

or, equivalently,

$$(T_1+nI)\ T_n(1)+BT_n(1)-(2n+1)T_n(1)+n^2T_{n-1}(1)=0$$

$$(T_1+B-n-1)T_n(1)+n^2T_{n-1}(1)=0 \tag{3.2.3}$$

But $T_1+B-(n+1)I=BA-nI$. Therefore relation (3.2.3) becomes

$$(BA-nI)(T_1+(n-1)I)y_{n-1}= -n^2y_{n-1}$$

or $\quad [BAT_1-nT_1+(n-1)BA+nI]y_{n-1}=0$

or $\quad B(AT_1-A+I)y_{n-1}= -(n-1)By_{n-1}$

$$BSy_{n-1}= -(n-1)By_{n-1}$$

which is true because y_{n-1} is an eigen vector for S of eigen value -(n-1).

3.3 SOME MORE THEOREMS ON ENDOMORPHISM:

The theorem (3.2.1) and Ly_n given by (3.1.3) allow the derivation of the following differential equation

$$C(x)a(x)y_n''+[c(x)a'(x)+c(x)b(x)+e(x)a(x)]y_n'+[c(x)b'(x)+e(x)b(x)-n]y_n=0 \tag{3.3.1}$$

Define $S= (A-I)BA$. Then $Sy_n= -ny_n$. We choose $c(x)=b(x)=0$. Then we can write the following differential equation .

$$e(x)a^2(x)y''+[e'(x)a(x)+a'(x)e(x)-e(x)]a(x)y'+ny=0 \tag{3.3.2}$$

We now introduce some special cases of the operators (3.1.1) which play a role analogous to that of the conventional special functions. Since most of the results of the paper will be concerned with the recurrence properties, Rodrigues type formulae and differential equations, it is worth mentioning the following special cases.

Case 1 : Let $V= C^\infty(R)$. $(1/a(x))=\frac{d}{dx}e(x)=e'(x)$, $b(x)=0$ $c(x)\neq 0$ in (3.1.1). Then operators A, B $\in$ EndV satisfy the commutation relation [A, B]=I. Thus in particular, if

$$(Af)x=\frac{x^{2-m}}{m(m-1)}f', \quad (Bf)x=-f'+mx^{m-1}f, \quad m\neq 0,\ 1 \tag{3.3.3}$$

then from (3.1.2), [A, B]=I

Case 2:

Let $V= C^\infty(R)$, $c(x)=b(x)=0$ and $e'(x)=1/a(x)$ in (3.1.1). Then we can apply (3.1.2) to check that operators $A,B\in$EndV satisfy the commutation relation [A, B]=I.Thus in particular, if

$$(Af)x = \frac{x^{1-m}}{m}f', \quad (Bf)x= x^m f, \quad m\neq 0, \text{ then}[A,B]=I$$

Theorem (3.3.1) (Generalized Rodrigues Formula)

$$(B^n f)x = (-1)^n e^{x^m} \{f(x)e^{-x^m}\}^{(n)}, \; m>1 \tag{3.3.4}$$

where $\{\ \}^{(n)}$ denotes the n^{th} differentiation of $\{\ \}$.

Proof:

For n=1, the above equality gives

$$(Bf)x = -e^{x^m} \{f'(x)e^{-x^m} - mx^{m-1}e^{-x^m}f(x)\} = -f'(x) + mx^{m-1}f(x), \tag{3.3.5}$$

which is the definition of B. Thus equality holds for n=1. Inductively for n+1, we have

$$(B^{n+1}f)x = (-1)^{n+1}e^{x^m}\{f(x)e^{-x^m}\}^{(n+1)}$$

Since

$$\begin{aligned}(B^{n+1}f)x &= B(B^n f)x \\ &= mx^{m-1}(B^n f)x - \{(B^n f)x\}^{(1)} \\ &= mx^{m-1}(-1)^n e^{x^m}\{f(x)e^{-x^m}\}^{(n)} - (-1)^{(n)}\{e^{x^m}\{f(x)e^{-x^m}\}^{(n)}\}^{(1)} \\ &= (-1)^{n+1} e^{x^m}\{f(x)\ e^{-x^m}\}^{(n+1)}\end{aligned}$$

which completes our proof.

For m=2, the above theorem reduces to a result of Radulescu ([91], p-68).

The operators A, B given in (3.3.3) may be combined with

(3.3.1) to yield the differential equation

$$x^{2-m}y''_n-[(m-2)x^{-m}+m]xy_n'+m(m-1)ny_n=0 \tag{3.3.6}$$

which is a generalization of Hermite differential equation [92]

$$y''-2xy'+2ny=0, n\in N \tag{3.3.7}$$

and can be obatined from (3.3.6) by setting m=2.

Set $y_0=1$ to obtain $y_n= B^n(1)$. Therefore defining $y_n= H_{m,n}(x)$ which is relevant to the well known Hermite polynomials $H_n(x)$, we get by theorem (3.3.1).

$$B^n(1)=(-1)^n e^{x^m}\frac{d^n}{dx^n}(e^{-x^m})=H_{m,n}(x) \tag{3.3.8}$$

for f(x)=1. Furthermore, for m=2, we easily find that (3.3.8) is the well known Rodrigues formula for Hermite polynomials $H_n(x)$.

Note that $H_{2,n}(x) = H_n(x)$

With the preceding technique it is possible to get the differential equation

$$x^{2-m}y_n''-(mx-x^{1-m})y_n'+m^2n\ y_n=0 \tag{3.3.9}$$

Starting from (3.3.2) and Case 2. For m=1 (3.3.9) reduces to Laguerre differential equation.

$$xy''+(1-x)y'+ny=0 \tag{3.3.10}$$

A noticeable feature of the theorem (3.3.1) is reported below. During the early 1930's Bell introduced a class of polynomials , named after him, denoted by $Y_n(x_1, x_2, \ldots\ldots\ldots x_n)$ and defined by the generating function

$$\exp\left(\sum_{i=1}^{\infty} \frac{x_i t^i}{s!}\right)=1+\sum_{n=1}^{\infty} \frac{t^n}{n!} Y_n(x_1,x_2,\ldots,x_n) \qquad (3.3.11)$$

Bell polynomials (B.P) has been used in physics, combinatorics, statistics and other fields and may perhaps be exploited in other research fields. They can be reduced to generalized Hermite, Gauss -Hermite and Bessel polynomials (See[38], [93]). According to Dattoli et al [38]

$$\left(\frac{d}{dx}\right)^m \exp(-x^m) = \exp(-x^m)$$

$$Y_{n,m}\left(-m\, x^{m-1}, -m(m-1)\, x^{m-2}, \ldots\ldots\ldots\ldots, -\frac{m!}{(m-n)!}\, x^{m-n}\right) \qquad (3.3.12)$$

where for n>m the arguments x_{m+n} are vanishing. Now using theorem (3.3.1) and (3.3.12), we get

$$B^n(1)=(-1)^n\, Y_{n,m}\left(-mx^{m-1}, -m(m-1)\, x^{m-2}, \ldots\ldots\ldots, -\frac{m!}{(m-n)!}\, x^{m-n}\right)$$

(3.3.13)

which can be viewed as an extended version of the Rodrigues formula. For m=2, we identify that

$$Y_{n,2}(-2x,\ -2) = (-1)^n\ H_n(x)$$

The richer structure of the operators given by (3.1.1) allows to introduce further operators.

$$A = \frac{1}{\sqrt{m}}\left(\frac{d}{dx}+x\right)$$

$$B = \sqrt{m}\left[x-\frac{1}{m^{m-1}}\left(\frac{d}{dx}+x\right)^{m-1}\right]$$

which satisfy the commutation relation [A, B] = I. In analogy to the ordinary case we introduce the harmonic oscillator like functions ([38] p- 600).

$$\phi_{n,m}(x)=\frac{1}{\sqrt{m^n n!}}\ H_{m,n}(x)\ \exp(-x^2/2)$$

where $H_{m,n}$ (x) are generalized Hermite polynomials defined by the generating function

$$\exp[\ mxt-\ t^m] = \sum_{n=0}^{\infty} \frac{H_{m,n}(x)}{n!}t^n$$

It is also easily proved that $\phi_{m,n}(x)$ satisfies the ordinary differential equation

$$x\frac{d}{dx}+x^2-\frac{1}{m^{m-1}}\left(\frac{d}{dx}+x\right)^m \ \phi_{m,n}(x)=n \ \phi_{m,n}(x)]$$

which for m= 2 reduces to the equation satisfied by the ordinary harmonic ascillator eigen functions , namely

$$\left(-\frac{1}{2}\frac{d^2}{dx^2}+\frac{x^2}{2}\right) \phi_{2,n}(x) =(n+1/2) \ \phi_{2,n}(x)$$

Hence, the complete algebraic structure relevant to harmonic ascillator eigen functions should be understood as that $\phi_{m,n}(x)$ is an eigen function with eigen value n of the operator

$$BA=x\frac{d}{dx}+x^2-\frac{1}{m^{m-1}}\left(\frac{d}{dx}+x\right)^m$$

which is self adjoint only for n=2. The consideration we have developed indicate that the introduction of more general classes of functions opens the possibility of speculating on generalized form of special functions and provides a further example of realization of the creation-annihilation operators.

The analysis is aimed at accounting for wealth of properties exhibited by Super - Gauss-Hermite functions (S.G.H.F)

$$G_{m,n}(x)=(-1)^n Y_{m,n}\left(-\left\{\frac{m!}{(m-s)!}x^{m-s}\right\}_n \ldots\ldots\right)\exp\left(\frac{-x^m}{2}\right),\ n\leq m$$

where m is even and larger than 2.

S.G.H.F. reduces to the usual Gauss - Hermite functions for m =2.

In analogy to the ordinary case, we now produce the operators A, B $\in$ EndV for V= $C^\infty(RxR)$ by

$$A= a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c$$

$$B=d\frac{\partial}{\partial x}+e\frac{\partial}{\partial y}+f \tag{3.3.14}$$

where a, b, c, d, e, f are functions of x and y. Then

$$[A,B]u=\frac{\partial u}{\partial x}(ad_x+bd_y-da_x-ea_y)+\frac{\partial u}{\partial y}(ae_x+be_y-db_x-eb_y)+u(af_x+bf_y-dc_x-ec_y) \tag{3.3.15}$$

The operators given by (3.3.14) have indeed been recognized as a crucial element to develop a complete point of view to the multivariable special functions. One of the most direct ways of exploring these special functions is to start from the following theorem of Mandal [76] (see also [22]).

Theorem (3.3.2)

Let A, B$\in$EndV such that [A, B] =0. We define the sequence $(Y_n)_n \subset V$ as follows : $AY_1=Y_0$ and $Y_{n+1}= BY_n$ for every $n \geq 0$. Then $AY_{n+1}=Y_n$ for every $n \geq 1$ and Y_n is an eigen vector of eigen value 1 for BA. For every $n \geq 1$.

Let u= $Y_n(x, y) = \phi_n(x)\, y^n \in C^\infty(R, R)$. Then for [A, B] =0 , the identity $BAY_n=Y_n$ holds using the operators (3.3.14) with theorem (3.3.2), we find that $\phi_n(x)$ satisfies the following differential equation.

$$a\, d\phi_n''+\phi_n'\left(da_x+\frac{nbd}{y}+cd+ea_y+\frac{nae}{y}+fa\right)+\phi_n\left(\frac{ndb_x}{y}+dc_x+\frac{n(n-1)eb}{y^2}\right.$$

$$\left.+\frac{enb_y}{y}+\frac{nce}{y}+ec_y+\frac{nbf}{y}+cf-1\right)=0 \qquad (3.3.16)$$

Case 3:

In particular, if we choose $a = x^2/y$, $b=-x$, $c=1/y$, $d=x^2y$, $e=xy^2$ $f=y(x+1)$ then (3.3.14) yields [A, B] =0 and (3.3.16) reduces to

$$x^2\, \phi_n''+(2x+2)\, \phi_n' -n(n+1)\, \phi_n=0 \qquad (3.3.17)$$

It is evident that the simple Bessel polynomial defined by Krall and Frink [64] is a solution of (3.3.17).

CASE 4:

Let $a=e^y$, $b=e^y/x=e$, $c=f=0$ and $d=e^{-y}$. Then $[A, B]=0$ and the operators A, B admits the function $f_n(x, y) = J_n(x)\, e^{ny}$ (where $J_n(x)$ is a Bessel function) as basis elements satisfying the recurrence relations

$$Af_n=f_{n+1}, \quad Bf_n=f_{n-1}, \quad ABf_n=BAf_n=fn.$$

The above relations (or equation (3.3.16)) immediately provide the differential equation obeyed by the Bessel functions, reported below for the sake of completeness

$$\left[-\frac{d^2}{dx^2}-\frac{1}{x}\frac{d}{dx}+\frac{n^2}{x^2}\right] J_n(x) = J_n(x) \tag{3.3.18}$$

The analysis presented here is definitely preliminary. Some analogies with the Lie-algebraic view to the ordinary Bessel functions (BF) theory can be recognized and usually exploited to infer further properties of the generalized Bessel function (GBF) and links with other special functions. Following Chiccoli et al[22], the following three variable extension of operators.

$$A= e^x \left(-\frac{\partial}{\partial y}+\frac{1}{y}\frac{\partial}{\partial z}+\frac{mt}{y}\frac{\partial}{\partial t}\right)$$

$$B= e^{-x} \left(\frac{\partial}{\partial y}+\frac{1}{y}\frac{\partial}{\partial z}+\frac{mt}{y}\frac{\partial}{\partial t}\right)$$

obey the required commutation rule [A, B]=0 and have the functions

$$f_n^{(m)} = f_n^{(m)} (x, y, z; t) = J_n^{(m)} (x, y; t) e^{nz}$$

as basis elements satisfying the relation $ABf_n^{(m)} = f_n^{(m)}$. Here $J_n^{(m)}$ is the GBF defined by

$$J_n^{(m)} (x, y; t) = \sum_{l=-\infty}^{\infty} t^l J_l(x) J_{n+ml}(y),$$

the parameter t being assumed complex GBF reduces to BF for x or y equal to zero, according to

$\lim\limits_{x\to 0} J_n^{(m)} = J_n(y)$ and $\lim\limits_{x\to 0} J_n^{(m)} = t^{-n/m} J_{-n/m}(x)$

for n/m integer otherwise zero.

The differential equation obeyed by GBF can be inferred from $Af_n^{(m)} = f_{n+1}^{(m)}$, $Bf_n^{(m)} = f_{n-1}^{(m)}$ and theorem (3.3.2). For the sake of illustration , we report the differential equation obeyed by GBF of two variables

$$\frac{\partial^2}{\partial x^2} J_n(x,y) = 4\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2}{\partial y^2}+1\right) J_n(x,y)$$

where $J_n^{(2)}(x,y; 1) = J_n(x, y)$.

CHAPTER 4

LIE-THEORETIC GENERATING FUNCTIONS OF MULTIVARIABLE GENERALIZED BESSEL FUNCTIONS

4.1 INTRODUCTION:

In this chapter, we discuss generating functions of multivariable generalized Bessel functions (GBF) by using a representation of the Lie group T_3. A few special cases of interest are also discussed.

G. Dattoli et al [20], [37] introduced and discussed a theory of generalized Bessel functions (GBF) in their standard form and their modified version. A set of functions which generalize the ordinary Bessel function (BF) and GBF was the topic of a recent study by C. Chicoli et al [18]. GBF are gaining more and more importance in physics since they play a crucial role in any scattering problem when the dipole approximation can not be used. The research activity was further stimulated by a number of problems in which GBF is an essential analytic tool [21].

The one-parameter generalized Bessel function $J_n(x,y;s)$ is defined as follows ([20], [37]).

$$J_n(x,y;s)= \sum_{l=-\infty}^{\infty} s^l J_{n-2l}(x) J_l(y) \qquad (x,y)\in R \tag{4.1.1}$$

and for s=1 equation (4.11) reduces to GBF $J_n(x,y)$ of two variables.

A simple calculation ([20], p-27 (1.16)) shows that $J_n(x,y;s)$

satisfies the following pure and differential recurrence relations:

$$\frac{\partial}{\partial s} J_n(x,y;s)= \frac{y}{2}[J_{n-2}(x,y;s) + \frac{1}{s^2} J_{n+2}(x,y;s)]$$

$$\frac{\partial}{\partial x} J_n(x,y;s)=½[J_{n-1}(x,y;s)- J_{n+1}(x,y;s)] \tag{4.1.2}$$

$$\frac{\partial}{\partial y} J_n(x,y;s)= ½[sJ_{n-2}(x,y;s)-\frac{1}{s} J_{n+2}(x,y;s)]$$

$$2nJ_n(x,y;s)=x[J_{n-1}(x,y;s)+J_{n+1}(x,y;s)]+2y[sJ_{n-2}(x,y;s)+\frac{1}{s}J_{n+2}(x,y;s)]$$

Due to noticeable importance of GBF for applications, it deserves some interest to get, in addition to the results obtained in [18], [37] and [20], further generating functions for these functions. The principal interest in our results lies in the fact that a number of special cases would yield inevitably to many new and known results of the theory of special functions. It is worth recalling that several of the fundamental identities of Miller ([80] p-62-63) for cylindrical functions and Graf's addition theorem ([44], p-64) are special cases of results discussed in section (4.2) and (4.3).

It should be noted that we will be primarily concerned with the representation theory of local Lie group T_3. However, it should be clear that this method can be generalized to higher-dimensional Lie algebras. The complete algebraic structure relevant to the GBF i.e. a five dimensional Lie -algebra, is provided in Chiccoli ([19], p-249). Lie algebra of the three dimensional Lie group $G_{p,q,}$ which forms a natural generalization of T_3 has been computed by Miller ([80], p-316) for obtaining a group-theoretic generalization of Bessel functions.

4.2 MULTIPLIER REPRESENTATION OF T_3 AND GENERATING FUNCTIONS:

Let τ_3 be the Lie algebra of a three - dimensional complex Lie group T_3, a multiplicative matrix group with elements ([80], p-10)

$$g(b,c,\tau) = \begin{pmatrix} 1 & 0 & 0 & \tau \\ 0 & e^{-\tau} & 0 & c \\ 0 & 0 & e^{\tau} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; b,c,\tau \in C \tag{4.2.1}$$

T_3 has the topology of C^3 and is simply connected ,

([89], ch-8) . A basis for τ_3 is provided by the matrices ([80], p-11).

$$j^{+} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad j^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$j^{3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.2.2}$$

with commutation relations

$$[j^3,j^+] = j^+,\ [j^3,j^-] = -j^-, \quad [j^+\ ,j^-\ ] = 0 \tag{4.2.3}$$

We introduce the first order linearly independent differential operators j^3 j^+ and j^- each of the form

$$A_1(x,y,z;s)\frac{\partial}{\partial x}+A_2(x,y,z;s)\frac{\partial}{\partial y}+A_3(x,y,z;s)\frac{\partial}{\partial z}+A_4(x,y,z;s)\ \frac{\partial}{\partial s}+A_5(x,y,z;s)$$

such that

$$\begin{aligned} j^3[J_n(x,y;s)\ e^{nz}] &= a_n J_n(x,y;s)\ e^{nz} \\ j^+[J_n(x,y;s)\ e^{nz}] &= b_n J_{n+1}(x,y;s)\ e^{(n+1)z} \qquad (4.2.4) \\ j^-[J_n(x,y;s)\ e^{nz}] &= c_n J_{n-1}(x,y;s)\ e^{(n-1)z} \end{aligned}$$

where a_n, b_n and c_n are expressions in n which are independent of x,y, z and s, where as each $A_i(x,y,z;s)$, i=1,2,3,4,5 is an expression in x, y, z and s which is independent of n.

Now using (4.2.4) and (4.1.2) , we get the following operators.

$$J^3 = \frac{\partial}{\partial z},\ J^{\pm} = e^{\pm z}\{\mp\ \frac{\partial}{\partial x}+\frac{1}{x}\ \frac{\partial}{\partial z}-\frac{2s}{x}\ \frac{\partial}{\partial s}\} \qquad (4.2.5)$$

These operators satisfy the commutation relations

$$[J^3,\ J^{\pm}] = \pm J^{\pm},\quad [J^+,\ J^-] = 0 \qquad (4.2.6)$$

According to theorem (1.4.1) the irreducible representation Q^{μ} (w, m_0) of τ_3 is defined for each μ,w, $m_0 \in C$ such that $w\neq 0$ and $0\leq \mathrm{Re}\, m_0<1$. The spectrum of J^3 is the set $S=\{m_0+n_0 : n_0$ an integer$\}$ and there is a basis $\{f_n\}$, $n\in S$ for the representation space V with the properties $J^3\ f_n = nf_n$, $Ef_n=\mu f_n$, $J^+\ f_n = wf_{n+1}$, $J^- f_n=wf_{n-1}$, $C_{0,0}f_n= (J^+J^-) f_n= w^2 f_n$ (4.2.7)

To construct a realization of $Q^{\mu}(w,\ m_0)$ in terms of the operators (4.2.5), we find non zero functions

$f_n(x,y,z;s) = z_n(x,y;s)\ e^{nz}$, $n\in S$

such that equations (4.2.7) are valid for all $n \in S$

Without any loss of generality for special function theory we can study only those representations $Q^{\mu}(w,m_0)$ for which $E=\mu=0$, $m_0=0$, $w=1$, $n=n_0$. Moreover, the representation $Q^0(1,0)$ of the 3-dimensional subalgebra τ_3. The action of Q(1, 0) is obtained explicitly from equations (4.2.5) and (4.2.7) by suppressing the operator E.

Expressed in terms of the functions $z_n(x,y;z)$ equations (4.2.7) become the recursion relations

$$\left[\frac{-\partial}{\partial x}-\frac{2s}{x}\frac{\partial}{\partial s}+\frac{n}{x}\right] z_n(x,y;s) = z_{n+1}(x,y;s)$$

$$\left[\frac{\partial}{\partial x}-\frac{2s}{x}\frac{\partial}{\partial s}+\frac{n}{x}\right] z_n(x,y;s) = z_{n-1}(x,y;s) \qquad (4.2.8)$$

$$\left[\frac{-\partial^2}{\partial x^2}+\frac{4s^2}{x^2}\frac{\partial^2}{\partial s^2}-\frac{1}{x}\frac{\partial}{\partial x}-\frac{4s}{x^2}(n-1)\frac{\partial}{\partial s}+\frac{n^2}{x^2}\right] z_n(x,y;s) = z_n(x,y;s)$$

where n is an integer.

If we choose

$Z_n(x,y;s)=J_n(x,y;s)$, $n \in S$

then the functions $f_n(x,y,z;s) = z_n(x,y;s)\, e^{nz}$ form an analytic basis for a realization of the representation Q(1,0) of τ_3. This representation of τ_3 can be extended to a local multiplier representation of T_3 by operators T(g), $g \in T_3$ on the space Cl of all functions analytic in a neighbourhood at the point $(x^0, y^0, z^0; s^0)= (1,0,0,0)$.

Following our usual method using ([80], p-18) and (4.2.5), we now proceed to compute the multiplier representation of T_3. The actions of the one parameter groups $\exp(\tau\, j^3)$, $\exp(cj^-)$ and $\exp(bj^+)$ are obtained by integrating the following differential equations,

$$\frac{dx(\tau)}{d\tau}=0,\frac{dy(\tau)}{d\tau}=0,\frac{dz(\tau)}{d\tau}=1,\frac{ds(\tau)}{d\tau}=0,\frac{dv(\tau)}{d\tau}=0$$

$$\frac{dx(c)}{dc}=e^{-z(c)},\frac{dy(c)}{dc}=0,\frac{dz(c)}{dc}=\frac{e^{-z(c)}}{x(c)},\frac{ds(c)}{dc}=\frac{-2s(c)e^{-z(c)}}{x(c)},\frac{dv(c)}{dc}=0$$

$$\frac{dx(b)}{db}=e^{z(b)},\frac{dy(b)}{db}=0,\frac{dz(b)}{db}=\frac{e^{z(b)}}{x(b)},\frac{ds(b)}{db}=\frac{-2s(b)e^{z(b)}}{x(b)},\frac{dv(b)}{db}=0 \tag{4.2.9}$$

subject to the conditions $x(0)= x^0$, $y(0)= y^0$, $z(0)= z^0$, $s(0)=s^0$, $\nu(0)=1$, where ν is the multiplier of the representation.

Hence, the values of the multiplier representations of $\exp(\tau j^3)$, $\exp(cj^-)$ and $\exp(bj^+)$ are given by

$$[T(\exp\tau j^3)f](x^0, y^0,t^0 ;s^0)=f(x^0, y^0, e^\tau t^0; s^0)$$

$$[T(\exp cj^-)f](x^0, y^0, t^0; s^0)=f[x^0 (1+ 2c/t^0x^0)^{1/2}, y^0,t^0 (1+2c/t^0x^0)^{1/2};$$
$$s^0(1+2c/t^0x^0)^{-1}] \tag{4.2.10}$$

$$[T(\exp bj^+)f](x^0, y^0, t^0; s^0)=f[x^0(1+2b\, t^0/x^0)^{1/2}, y^0,t^0(1-2b\, t^0/x^0)^{-1/2} ;$$
$$s^0(1-2b\, t^0/x^0)]$$

where $t^0 = e^{z^0}$

If $g\in T_3$ is given by (4.2.1), we find

$g=\exp(bj^+)\ \exp(cj^-)\ \exp(\tau j^3)$

and

$$[T(g)f](x,y,t;s)=T[(\exp bj^{+})\ (\exp cj^{-})\ (\exp\tau j^{3})f]\ (x,y,t;s) \tag{4.2.11}$$

$$=[T(\exp bj^{+})T(\ \exp cj^{-})\ T(\exp\tau j^{3})f](x,y,t;s)$$

A straight forward computation using (4.2.10) and (4.2.11) yields

$$[T(g)f]\ (x,y,t;s)=\ f[x\sqrt{\theta\phi},\ y,\ te^{\tau}\sqrt{\frac{\phi}{\theta}},\ \frac{s\theta}{\phi}] \tag{4.2.12}$$

defined for $|2bt/x| <1$, $|2c/tx|<1$ where

$\theta= 1-2bt/x,$ $\phi=1+2c/\ tx$

According to Miller ([80], section 2.2), our realization of the representation Q(1, 0) of τ_3 on the space generated by the function $f_n(x,y,z;s)$, $n\in S$, can be extended to a local representation T_3, where the group action is given by (4.2.12).

The matrix elements of the local representation with respect to the basis $f_n(x,y,z;s) =J_n(x,y;s)e^{nz}$ are uniquely determined by Q(1,0), and we obtain the relations

$$[T(g)f_k](x,y,t;s)=\ \sum_{l=-\infty}^{\infty} A_{lk}(g)f_l\,(x,y,t;s) \tag{4.2.13}$$

where $k= 0,\ \pm1,\ \pm2,\dots\dots\dots$

$$t^k\ e^{k\tau}\left(\frac{\phi}{\theta}\right)^{\frac{k}{2}} J_k\ [x\sqrt{\theta\phi},\ y;\ \frac{s\theta}{\phi}]=\ \sum_{l=-\infty}^{\infty} A_{lk}(g)J_l\,(x,y;s)t^l \tag{4.2.14}$$

where $\theta=1-\ 2bt/x$, $\phi=1+2c/tx$ and the matrix elements $A_{lk}(g)$ are given by ([80], p-56(3.12))

$$A_{lk} = \frac{e^{k\tau}\ (c)^{(k-l+|k-l|)/2}\ (b)^{(l-k+|k-l|)/2}}{|k-l|!}\ {}_0F_1(|k-l|+1;bc) \qquad (4.2.15)$$

valid for all integral values of l and k.

since $J_n(x,y;s)$, $n \in S$, is analytic in x,y and s for all non-zero values of x,y and s, the infinite series (4.2.14) converges absolutely for $|2bt/x| < 1$, $|2c/tx| < 1$. Thus our main generating function becomes

$$\left(\frac{\phi}{\theta}\right)^{\frac{k}{2}} J_k[x\sqrt{\theta\phi},\ y;\ \frac{s\theta}{\phi}] =$$

$$\sum_{n=-\infty}^{\infty} \frac{c^{-(n+|n|)/2} b^{(n+|n|)/2}}{|n|!}\ {}_0F_1(|n|+1;bc)\ J_{k+n}(x,y;s)t^n \qquad (4.2.16)$$

defined for $|2bt/x| < 1$, $|2c/tx| < 1$, where

$\theta = 1 - 2bt/x$, $\phi = 1 + 2c/tx$

If $bc \neq 0$, we can introduce the coordinates r, ν defined by $r = (ibc)^{1/2}$ and $\nu = (b/ic)^{1/2}$ such that $b = r\nu/2$, $c = -r/2\nu$. In this case equation (4.2.16) yields the generating function

$$(\beta/\alpha)^{k/2}\ j_k[\ x\sqrt{\alpha\beta},\ y;\ s\alpha/\beta] = \sum_{n=-\infty}^{\infty} (-\nu)^n J_n(-r) J_{k+n}(x,y;s) t^n$$

where $\alpha = 1 - r\nu t/x$, $\beta = 1 - r/\nu tx$, $|r\nu t/x| < 1$, $|r/\nu tx| < 1$ (4.2.17)

4.3 SPECIAL CASES:

We shall mention a few special cases of (4.2.16).

I. If c=0, t=1, equation (4.2.16) becomes

$$(1-2b/x)^{-k/2} J_k[x(1-2b/x)^{½}, y;s(1-2b/x)] \tag{4.3.1}$$
$$= \sum_{n=0}^{\infty} \frac{b^n}{n!} J_{k+n}(x,y;s), \quad |2b/x|<1$$

If b=0, t=1, equation (4.2.16) becomes,

$$(1+2c/x)^{k/2} J_k[x(1+2c/x)^{½}, y;s(1+2c/x)^{-1}] \tag{4.3.2}$$

$$= \sum_{n=0}^{\infty} \frac{c^n}{n!} J_{k-n}(x,y;s), \quad |2c/x|<1$$

II. Let us consider the limiting case

$$J_n(x,0;s)= J_n(x) \tag{4.3.3}$$

Making use of (4.3.3) in (4.2.16) equation (4.2.16) reduces to special case of Miller([80], p-62(3.29))

$$[1-2bt/x]^{-k/2} [1+2c/tx]^{k/2} J_k[x(1-2bt/x)^{1/2}(1+2c/x)^{1/2}]$$

$$= \sum_{n=-\infty}^{\infty} \frac{c^{-(n+|n|)/2} b^{(n+|n|)/2}}{|n|!} {}_0F_1(|n|+1;bc) \; J_{k+n}(x)t^n \tag{4.3.4}$$

where $|2bt/x|<1$, $|2c/tx|<1$

If $bc \neq 0$, we introduce the coordinates r, v defined by $r=(ibc)^{½}$ and $v= (b/ic)^{1/2}$ such that $b=rv/2$, $c=-r/2v$.

In this case, equation (4.3.4) yields the generating function

$$[1-rvt/x]^{-k/2} [1-r/vtx]^{k/2} J_k[x(1-rvt/x)^{1/2}(1-r/vtx)^{1/2}]$$

$$= \sum_{n=-\infty}^{\infty} (-v)^n J_n(-r) J_{k+n}(x) t^n, \quad |rvt/x|<1, \quad |r/vtx|<1 \tag{4.3.5}$$

setting t=1, in (4.3.5) we get

$$[1-r\nu/x]^{-k/2}\ [1-r/\nu x]^{k/2}\ J_k[x(1-r\nu/x)^{1/2}(1-r/\nu x)^{1/2}]$$

$$=\sum_{n=-\infty}^{\infty}(-\nu)^n J_n(-r)J_{k+n}(x),\quad |r\nu/x|<1,\quad |r/\nu x|<1 \qquad (4.3.6)$$

which is a generalization of Graf's addition theorem ([44] p-44).

We shall now obtain few special cases of (4.3.4).

If c=0, t=1, equation (4.3.4) reduces to special case of Miller ([80], p-62(3.30)]

$$(1-2b/x)^{-k/2}\ J_k[x(1-2b/x)^{1/2}] = \sum_{n=0}^{\infty}\frac{b^n}{n!}\ J_{k+n}(x),\quad |2b/x| <1 \qquad (4.3.7)$$

If b=0, t=1, equation (4.3.4) reduces to special case of Miller ([80], p-62(3.31)]

$$(1+2c/x)^{k/2}\ Jk[x(1+2c/x)^{1/2}] = \sum_{n=0}^{\infty}\frac{c^n}{n!}\ J_{k-n}(x),\quad |2c/x| <1 \qquad (4.3.8)$$

CHAPTER-5

SPECIAL LINEAR GROUP AND GENERATING FUNCTIONS

5.1 INRTRODUCTION:

In this chapter we will discuss a method for obtaining certain generating functions by using Lie theoretic method. This process given in a paper of S. Jain and H. L. Manocha [59] involves linear differential operators which form a three dimensional Lie algebra which is isomorphic to the Lie algebra $sl(2)$([80] p-8). Depending on these operators we will determine local multiplier representation $[T(g)f](x,y)$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2).$$

This multiplier representation leads to generating functions for Laguerre functions after choosing $f(x,y)$ in certain ways. As an application , a generating function involving hypergeometric function of one and two variables and Laguerre function is obtained. Several special cases of the main generating function are also obtained.

5.2 DIFFERENTIAL OPERATORS AND MULTIPLIER REPRESENTATION :

We have the differential equation ([92], p-204)

$$x\frac{d^2u}{dx^2}+(1+\gamma-x)\frac{du}{dx}+(m+n)u=0 \tag{5.2.1}$$

The solution of (5.2.1) is given by

$$u(x)=L^{(\gamma)}_{m+n}(x) \tag{5.2.2}$$

By substituting $y\frac{\partial}{\partial y}$ for n in (5.2.1), we construct the following partial differential equation

$$\left[x\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+m\right]f(x,y)=0 \tag{5.2.3}$$

$f(x, y)=y^n u(x)$ is a solution of (5.2.3)

Now, we introduce the first order partial differential operators

$$j^3= y\frac{\partial}{\partial y}+m+(\gamma+1)/2$$

$$J^+=xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+(m+\gamma+1-x)y \tag{5.2.4}$$

$$J^- = xy^{-1}\frac{\partial}{\partial x}-\frac{\partial}{\partial y}-my^{-1}$$

These operators obey the following commutation relations

$$[J^3,J^\pm]=\pm J^\pm,\quad [J^+, J^-]= 2J^3 \tag{5.2.5}$$

These J- operators form the basis of a Lie algebra *sl*(2)[80].

The Casimir operator

$$C= J^+J^-+J^3J^3-J^3=\left[x^2\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+mx+\frac{\gamma^2-1}{4}\right] \tag{5.2.6}$$

commutes with J^3, J^+ and J^-, (5.2.3) may be rewritten as

$$Cf(x,y)=\frac{\gamma^2-1}{4}f(x,y) \tag{5.2.7}$$

To determine the multiplier representation induced by J operators, we need to compute the expressions ([80], p-3) $e^{a'J^-}\ e^{b'J^+}\ e^{c'J^3}$

$$e^{a'J^-}\ e^{b'J^+}\ e^{c'J^3}\ f(x,\ y) = \exp\left\{\frac{(m+\gamma+1)c'}{2}\right\} \times \exp\left\{\frac{-b'xy}{1+a'b'-b'y}\right\}$$

$$\times\ (1+a'b'-b'y)^{-(\gamma+1+m)}\left(1-\frac{a'}{y}\right)^{m}$$

$$\times\ f\left(\frac{xy}{(y-a')(1+a'b'-b'y)}, \frac{y-a'}{1+a'b'-b'y}e^{c'}\right) \tag{5.2.8}$$

The complex parameters a', b' and c' are related to $g \in SL(2)$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc=1 \text{ by ([80], p-8)}$$

$e^{c'/2}=a$, $a' = -c/a$, $b'=-ab$

Therefore, for g in a sufficiently small neighbourhood of the identity element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL(2)$

$$[T(g)f](x,y)=\exp\left(\frac{bxy}{d+by}\right)(d+by)^{-(\gamma+1+m)}(a+c/y)^{m} \times f\left(\frac{xy}{(d+by)(c+ay)}, \frac{c+ay}{d+by}\right) \tag{5.2.9}$$

$|by/d| < 1$, $-\pi < \arg a,\ \arg d < \pi$, $ad-bc=1$

5.3 GENERATING FUNCTIONS:

A. We choose f(x,y) to be a common eigen function of the operators C and $J^3J^3 + (\gamma'-\gamma-2m-1)J^3 - J^+$

Let f(x,y) satisfy the simultaneous equations

$$Cf(x,y) = \frac{\gamma^2-1}{4} f(x,y),$$

$$[J^3J^3+(\gamma'-\gamma-2m-1)J^3-J^+]f(x,y)=\frac{(m+\gamma+1)}{2}(\gamma'-m-(\gamma+1)/1)f(x,y) \tag{5.3.1}$$

which may be rewritten as

$$\left[x\frac{\partial^2}{\partial x^2}+(\gamma+1-x)\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-(m+\gamma+1)\right]e^x\ f(-x,y)=0$$

$$\left[y\frac{\partial^2}{\partial y^2}+(\gamma'+1-y)\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}-(m+\gamma+1)\right]e^x\ f(-x,y)=0 \tag{5.3.2}$$

These equations have solution ([43], p-234)

$$e^x\ f(-x,y)= \psi_2(m+\gamma+1,\ \gamma+1,\ \gamma'+1;\ x,y) \tag{5.3.3}$$

where $\psi_2(\alpha;\ \gamma,\ \gamma';x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} x^m y^n}{m!n!(\gamma)_m(\gamma')_n}$

We rewrite (5.3.3) as

$$f(x,y)= e^x\psi_2\ (m+\gamma+1,\ \gamma+1,\ \gamma+1;\ -x,y) \tag{5.3.4}$$

Therefore

$$[T(g)f](x,y)= \exp\left(\frac{axy}{c+ay}\right)\ (d+by)^{-(\gamma+1+m)}\ (a+c/y)^m$$

$$\psi_2\ [m+\gamma+1,\ \gamma+1,\ \gamma'+1;\ \frac{-xy}{(c+ay)(d+by)},\frac{c+ay}{d+by}] \tag{5.3.5}$$

$|by/d| < 1$, $-\pi < \arg a$, $\arg d < \pi$, $ad-bc=1$

$[T(g)f](x,y)$ satisfies

$$C[T(g)f](x,y) = \left(\frac{\gamma^2-1}{4}\right) [T(g)f](x,y) \tag{5.3.6}$$

(5.3.5) has an expansion of the form

$$[T(g)f](x,y) = \sum_{n=-\infty}^{\infty} k_n(g) L_{m+n}^{(\gamma)}(x) y^n \tag{5.3.7}$$

Putting x=0, this gives

$$k_n(g) = a^m (-b)^n d^{-(\gamma+m+n+1)}$$

$$\frac{\Gamma(1+m+n)}{(1+\gamma)_m} \sum_{p=0}^{\infty} \frac{(-m)_p (1+\gamma+m+n)_p}{p!\Gamma(1+n+p)} \left(\frac{bc}{ad}\right)^p {}_2F_2\left[\begin{matrix} -n-p,\ 1+m \\ \gamma'+1,\ 1+m-p \end{matrix}; a/b\right] \tag{5.3.8}$$

Thus the generating function (5.3.7) becomes

$$\exp\left(\frac{axy}{c+ay}\right)\left(1+\frac{by}{d}\right)^{-(\gamma+1+m)}\left(1+\frac{c}{ay}\right)^m$$

$$\psi_2\left[m+\gamma+1,\ \gamma+1,\ \gamma'+1;\ \frac{-xy}{(c+ay)(d+by)},\ \frac{c+ay}{d+by}\right]$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(1+m+n)(-b/d)^n y^n}{(1+\gamma)_m} L_{m+n}^{(\gamma)}(x) \sum_{p=0}^{\infty} \frac{(-m)_p (1+\gamma+m+n)_p}{p!\Gamma(1+n+p)} {}_2F_2\left[\begin{matrix} -n-p, 1+m \\ \gamma'+1, 1+m+p \end{matrix}; \frac{a}{b}\right],$$

$$|c/ay| < 1, \quad |by/d| < 1, \quad ad-bc=1 \tag{5.3.9}$$

where the terms comprising to n= -1, -2, -3,.. are well defined because of the relation

$$\underset{\mu\to k}{Lt}\sum_{p=0}^{\infty}\frac{(1+\gamma+m+\mu)_p(-m)_p(bc/da)^p}{p!\Gamma(1+\mu+p)}{}_2F_2\left[\begin{matrix}-\mu-p,-1-\gamma-m\\ \gamma'+1,1+m-p\end{matrix};\frac{a}{b}\right]$$

$$=\frac{(1+\gamma+m-k)_k(-m)_k}{k!}\left(\frac{bc}{ad}\right)^k\sum_{p=0}^{\infty}\frac{(1+\gamma+m)(-m+k)_p(bc/ad)^p}{p!(1+k)_p}{}_2F_2\left[\begin{matrix}-p,1+m\\ \gamma'+1,1+m-p-k\end{matrix};\frac{a}{b}\right]$$

$$k=1,2,\ldots\ldots \tag{5.3.10}$$

SPECIAL CASES:

(5.3.9) gives the following special cases:

$$\frac{-xy}{e^{w-y}}\left(1-\frac{w}{y}\right)^m\psi_2(m+\gamma+1,\gamma+1,\gamma'+1;\frac{xy}{w-y},w-y)$$

$$=\sum_{n=-\infty}^{\infty}\frac{(1+n)_m}{(1+\gamma)_m(1+\gamma')_n}L_{m+n}^{(\gamma)}(x)\,{}_2F_2\left[\begin{matrix}1+\gamma+m+n,1+\gamma'-\gamma+n\\ 1+n,\gamma'+1+n\end{matrix};w\right] \tag{5.3.11}$$

and where the terms corresponding to n= -1,-2,.......are well defined because of the relation of the type (5.3.10);

$e^x\ (1-y)^{-(\gamma+1+m)}\ \psi_2[m+\gamma+1,\ \gamma+1,\ \gamma'+1;\ -x/1-y,\ -wy/(1-y)\ )$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(1+n+m)}{(1+\gamma)_m(1+\gamma')_n}L_{m+n}^{(\gamma)}(x)L_n^{(\gamma')}(w)y^n \tag{5.3.12}$$

B. We choose f(x,y) to be a common eigen function of the

operator C and

$$J^-J^3-\left(m+\frac{\gamma+1}{-2}\right)J^-J^3$$

Let f(x,y) satisfy the simultaneous equation

$$Cf(x,y)=\left(\frac{\gamma^2-1}{4}\right) f(x,y)$$

$$[J^-J^3-(m+((\gamma+1)/2)J^- - J^3]\ f(x,y)=0 \tag{5.3.13}$$

which may be rewritten as

$$\left[x\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+m\right]f(x,y)=0 \tag{5.3.14}$$

$$\left[-y\frac{\partial^2}{\partial y^2}+\frac{x\partial^2}{\partial x\partial y}-(1+m+y)\frac{\partial}{\partial y}-\left(m+\frac{\gamma+1}{2}\right)\right]f(x,y)=0$$

These equations have a solution ([43], 235)

$$f(x,y)=H_4[-m,\ m+(\gamma+1)/2;\ 1+\gamma;\ x,y) \tag{5.3.15}$$

where $H_4[\alpha,\ \gamma,\ \delta;\ x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\gamma)_n}{m!n!(\delta)_m} x^m y^n$

Therefore,

$$[T(g)f](x,y)=\exp\left(\frac{bxy}{d+by}\right)\ (d-by)^{-(\gamma+1+m)}\ (a+c/y)^m$$

$$\times\ H_4[-m,\ m+(\gamma+1)/2;\ 1+\gamma;\ \frac{xy}{(c+ay)(d+by)},\ \frac{c+ay}{d+by}] \tag{5.3.16}$$

$|by/d|<1,\quad -\pi<\arg a,\ \arg d<\pi,\ ad-bc=1$

By the similar analysis as we did in section A. We obtain the generating function

$$\exp\left(\frac{bxy}{d+by}\right)\ (1+by/d)^{-(\gamma+1+m)}\ (1+c/ay)^{m}$$

$$\times\ H_4[-m,\ m+(\gamma+1)/2;\ 1+\gamma;\ \frac{xy}{(c+ay)(d+by)},\ \frac{c+ay}{d+by}]$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(1+m+n)}{(1+\gamma)_m}\left(\frac{-by}{d}\right)^n L_{m+n}^{(\gamma)}(x) \sum_{p=0}^{\infty} \frac{(1+\gamma+m+n)_p(-m)_p(bc/da)^p}{p!\Gamma(1+n+p)}$$

$$\times\ {}_2F_2\left[\begin{matrix} -n-p, m+\frac{\gamma+1}{2} \\ 1+\gamma+m, 1+m-p \end{matrix};-\frac{a}{b}\right], \tag{5.3.17}$$

$|c/ay|<1$, $|by/d|<1$, $ad-bc=1$ where the terms corresponding to n= -1,-2,........... are well defined because of the relation (5.3.10).

SPECIAL CASES:

$$e^{(-xy/1-y)}\ (1-y)^{-(1+\gamma+m)}\ H_4[-m,\ m+(\gamma+1)/2;\ 1+\gamma;\ x/1-y;-wy/1-y]$$

$$= \sum_{n=-\infty}^{\infty} \frac{(1+n)_m}{(1+\gamma)_m} L_{m+n}^{(\gamma)}(x)\, {}_2F_2\left[\begin{matrix} -n, m+\frac{\gamma+1}{2} \\ m+1, 1+m+\gamma \end{matrix};w\right] y^n \tag{5.3.18}$$

where the terms corresponding to n= -1, -2,..... are well defined because of the relation of the type (5.3.10)

$$(1-w/y)^m\ H_4[\ -m,\ m+(\gamma+1)/2\ ;\ 1+\gamma;\ xy/w-y,\ w-y]\ e^{w}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(1+m)}{(1+\gamma)_m} L_{m+n}^{(\gamma)}(x) L_{m+(\gamma-1/2)}^{(n)}(w) y^n \tag{5.3.19}$$

C. We choose f(x,y) to be a common eigen function of the operator C and

$$-J^-J^3+\left(\frac{m+\gamma+1}{2}\right)\ J^-\ +J^3$$

Let f(x,y) satisfy the simultaneous equation

$$Cf(x,y)\ =\ \left(\frac{\gamma^2-1}{4}\right)\ f(x,y) \tag{5.3.20}$$

$$\left(-J^-J^3+\left(\frac{m+\gamma+1}{2}\right)J^-+J^3\right)\ f(x,y)\ =\ \left(\frac{m+\gamma-1}{2}\right)f(x,y)$$

This may be written as

$$\left[x\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+m\right]f(x,y)=0 \tag{5.3.21}$$

$$\left[y\frac{\partial^2}{\partial y^2}-\frac{x\partial^2}{\partial x\partial y}+(1+m+y)\frac{\partial}{\partial y}+1\right]f(x,y=0)$$

These equations have a solution ([43], 235)

$$f(x,y)=\ H_5[-m;\ 1+\gamma;\ x,y] \tag{5.3.22}$$

where

$$H_5[\alpha,\delta;\ x,y]\ =\ \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}}{m!n!(\delta)_m}x^m y^n$$

The generating function which we now obtain is

$$\exp\left(\frac{bxy}{d+by}\right)\left(1+\frac{by}{d}\right)^{-(\gamma+1+m)}\left(1+\frac{c}{ay}\right)^m H_5\left[-m;1+\gamma;\frac{xy}{(c+ay)(d+by)},\frac{c+ay}{d+by}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+m+n)}{(1+\gamma)_{m+n}}\left(\frac{-by}{d}\right)^{n}L_{m+n}^{(\gamma)}(x)\sum_{p=0}^{\infty}\frac{(-m)_p}{p!}\left(\frac{bc}{ad}\right)^{p}L_{n+p}^{(\gamma+m)}\left(-\frac{a}{b}\right) \qquad (5.3.23)$$

where the terms corresponding to n=-1,-2, are well defined because of the relation of the type (5.3.10).

SPECIAL CASES:

$$exp\left(\frac{-xy}{1-y}\right)(1-y)^{-(\gamma+1+m)}H_5[-m;1+\gamma;\frac{x}{1-y},\frac{-wy}{1-y}]$$

$$=\sum_{p=0}^{\infty}\frac{\Gamma(1+m+n)}{(1+\gamma)_{m+n}}L_{m+n}^{(\gamma)}(x)L_{n}^{(\gamma+m)}(w)y^{n} \qquad (5.3.24)$$

and $\left(1-\frac{w}{y}\right)^{m}H_5\left[-m;1+\gamma;\frac{-xy}{w-y},w-y\right]=$

$$\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+m)}{\Gamma(1+n)(1+\gamma)_m}L_{m+n}^{(\gamma)}(x)\,{}_1F_1[-m;1+n;-w]y^{n} \qquad (5.3.25)$$

D. We choose f(x,y) to be a common eigen function of the operator C and

$[J^3J^3 - J^-J^+ + (m+(\gamma+1)/2)J^- - (m+1)J^3]$

Let f(x,y) satisfy the simultaneous equations

$$Cf(x,y)=\left(\frac{\gamma^2-1}{4}\right)f(x,y) \qquad (5.3.26)$$

$[J^3J^3 - J^-J^3 + (m+(\gamma+1)/2)J^- - (m+1)J^3]\ f(x,y)=0$

which may be rewritten as

$$\left[x\frac{\partial^2}{\partial x^2}+(1+\gamma-x)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+m\right]f(x,y)=0 \qquad (5.3.27)$$

$$\left[y(y+1)\frac{\partial^2}{\partial y^2}-\frac{x\partial^2}{\partial x\partial y}+\{1+m+(1+m+\gamma)y\}\frac{\partial}{\partial y}+\left(\frac{\gamma-1}{2}\right)\left(m+\frac{\gamma+1}{2}\right)\right]\mathrm{f(x,y=0)}$$

These equations have a solution ([43], 235)

$$f(x,y) = H_{11}[-m,\ \frac{\gamma-1}{2},\ m+\frac{\gamma+1}{2};\ 1+\gamma;\ x,y] \tag{5.3.28}$$

where

$$H_{11}(\alpha,\beta,\gamma;\delta;x,y)= \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_n(\gamma)_n}{m!n!(\delta)_m}x^m y^n,\ |y|< 1$$

Thus, we get a generating function

$$\exp\left(\frac{bxy}{d+by}\right)\left(1+\frac{by}{d}\right)^{-(\gamma+1+m)}\left(1+\frac{c}{ay}\right)^m$$

$$\times\ H_{11}\left[-m,\frac{\gamma-1}{2},m+\frac{\gamma+1}{2};\frac{xy}{(c+ay)(d+by)},\frac{c+ay}{d+by}\right]$$

$$=\sum_{n=-\infty}^{\infty}\frac{\Gamma(1+m+n)}{(1+\gamma)_m}\left(\frac{-by}{d}\right)^n L_{m+n}^{(\gamma)}(x)$$

$$\times\ \sum_{p=0}^{\infty}\frac{(-m)_p(1+m+n+\gamma)_p}{p!\Gamma(1+n+p)}\left(\frac{bc}{ad}\right)^p {}_3F_2\left[\begin{matrix}\frac{\gamma-1}{2},m+\frac{\gamma+1}{2},-n-p\\ m+1-p,1+\gamma+m\end{matrix};\frac{a}{b}\right] \tag{5.3.29}$$

$$|c/ay|<1,\ |by/d|<1,\ ad-bc=1$$

where the terms corresponding to n=-1, -2,........ are well defined because of the relation of the type (5.3.10)

SPECIAL CASES:

$$exp\left(\frac{xy}{1-y}\right)(1-y)^{-(\gamma+1+m)}H_{11}[-m,\frac{\gamma-1}{2},m+\frac{1+\gamma}{2};1+\gamma;\frac{x}{1-y},\frac{wy}{1-y}]$$

$$= \sum_{n=-\infty}^{\infty} \frac{(1+n)_m}{(1+\delta)_m} L_{m+n}^{(\gamma)}(x)\, {}_3F_2\left[\begin{matrix} \frac{\gamma-1}{2}, m+\frac{\gamma+1}{2}, -n; w \\ m+1; 1+m+\gamma \end{matrix}\right] y^n \tag{5.3.30}$$

and

$$\left(1-\frac{w}{y}\right)^m H_{11}[-m, \frac{\gamma-1}{2}, m+\frac{1+\gamma}{2}; 1+\gamma; \frac{-xy}{w-y}, w-y]$$

$$= \sum_{n=-\infty}^{\infty} \frac{\Gamma(1+m)\left(\frac{\gamma-1}{2}\right)_n \left(m+\frac{\gamma+1}{2}\right)_n}{\Gamma(1+n)(1+\gamma)_m} L_{m+n}^{(\gamma)}(x)$$

$$\times\ {}_2F_1\left[\begin{matrix} m+n+\frac{\gamma-1}{2}, m+n+\frac{\gamma+1}{2}; -w \\ 1+n \end{matrix}\right] y^n \tag{5.3.31}$$

where the terms corresponding to n= -1, -2,........ are well defined by a relation of the type (5.3.10).

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